A PARENTS' GUIDE TO BASES

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As Tom Lehrer once said, "Base eight is just like base ten, if you're missing two fingers", so don't panic, this is easy. We'll go through a view of our usual expression of numbers in decimal form (base ten) just to get started . Then, we'll do the example of base eight and how to change back and forth between base eight and base ten, and go on to doing simple arithmetic in base eight. At the end, we'll cover other bases, such as base 2 and base 16.

Why do we tell our kids about bases other than the usual base ten? Base 2, 8 and 16 are used in computers, as we will explain. Also, working in other bases helps solidfy their understanding of place-value and methods of arithmetic in base ten. Finally, other bases give an introduction to the very important area of *modular arithmetic*, which is one of the most basic techniques used in applications of mathematics to things like encoding data, digital signal processing and other important processes.

I don't have the answers for the practice problems yet, I'll hand these out next week. Or, you can look for them at my home-page for Math League:

www.math.neu.edu/˜levine/MathLeague/home.html

1. Base ten

As you know, we write all numbers using only the digits $0, 1, \ldots, 9$, by using *places* to give the same digit different values. For example, in 492, the 4 means 4 hundreds, the 9 means 9 tens and the 2 just means 2.

But why ones, tens and hundreds? If you think of a number as currency, this is clear: suppose we only have one dollar, ten dollar, and hundred dollar bills. Then you can have any amount of money from \$0 to \$999 with at most 9 of each type of bill. Once you have 10 or more ones, you can change 10 of them in for 1 ten; once you have 10 or more tens, you can change 10 of them in for a hundred.

To keep this up, the next denomination you would need would be 10 hundreds, or one thousand, and so on. So, getting back to numbers, you start with the one's place, up to 9, the tenth one you change to 1 in the ten's place. After a 9 in the ten's place, you go to 10 tens, $10 \times 10 = 100$, after 9 100's you go to $10 \times 100 = 1000$ and so on.

2. Base eight

Now suppose you are missing two fingers: we are only allowed to use the digits $0, 1, \ldots, 7$. Going to the currency idea, suppose you travel to a country which also uses dollars as their currency, but they like 8s: they have 1 dollar bills, 8 dollar bills, 64 dollar bills, etc. This will work for them, and in fact, each sum of money (up to a certain amount) can be achieved with at most 7 of each type of bill: once you have 8 or more ones, change 8 of them in for an 8 dollar bill, once you have 8 or more 8 dollar bills, change them in for a 64 dollar bill.

Question: what is the next higher denomination bill?

Now you arrive with your money in 1s, ten's and hundred's and you have to exchange your \$492. How many 1s, 8s and 64s will you get?

Of course, you could just get 492 ones, pile them into groups of 8, exchange each pile for an 8 dollar bill, pile these into groups of 8 and exchange each pile of eight 8's for a 64 dollar bill. This would work, but you are sure to make a mistake, and it is rather tedious.

Instead, work from the top down. How many 64 dollar bills can you get for \$492:

$$492 \div 64 = 7 \text{ with a remainder of } 44$$

because $7 \times 64 = 448$ and $448 + 44 = 492$. So, you can get 7 sixty four dollar bills and you have \$44 left. For your \$44, you have

$$44 \div 8 = 5 \text{ with a remainder of } 4$$

$(5 \times 8 + 4 = 44)$ so you get 5 eight dollar bills and you have 4 ones left over. So: 7 64 dollar bills, 5 eight dollar bills and 4 one dollar bills = \$492.

In math terminology, this is

$$492_{10} = 754_8$$

and now you know all about base 8.

Perhaps we would like to deal with larger numbers base eight, so we have to go beyond the 64s. You can see the pattern, you exchange 8 64 dollar bills for a $8 \times 64 = 512$ bill, eight 512 bills for an $8 \times 512 = 4096$ bill, etc. The denominations are just the powers of 8:

$$8^0 = 1, 8^1 = 8, 8^2 = 64, 8^3 = 512, 8^4 = 4096$$

just as the denominations for base ten are the powers of ten.

After an enjoyable vacation in the land of 8 dollar bills, you are ready to go home. You need to change your money back. You have spent some, and you now have \$$546_8$ left, that is, 5 sixty-fours, 4 eights and 6 ones. How much is that in real money? This is easier, since we are used to doing arithmetic in base ten (I'll leave off the $_{10}$ when working with our usual base-ten numbers)

$$\begin{aligned} 546_8 &= 5 \times 64 + 4 \times 8 + 6 \\ &= 320 + 32 + 6 = 358 \end{aligned}$$

Practice problems Convert the base ten numbers to base eight, and the base eight numbers to base ten (answers coming in the next sheet).

$$1)45_8,\ 2)743_8,\ 3)3247_8,\ 4)746_8$$

$$5)93,\ 6)127,\ 7)362.$$

3. Arithmetic base eight

You can also add, subtract, multiply and divide in base eight. One way to do this would be to convert to base ten, do the arithmetic and then reconvert back to base eight. This would of course work, but it is too much trouble.

Instead, just think of the currency model: if you have 5 eights and 3 ones, and you win \$$67_8$, how much do you have? Well, the 7 ones and the 3 ones make ten, which is one eight plus 2 ones. The 5 eights and 6 eights make 11 eights, plus the 1 eight you got from the ones makes 12 eights, and this you can exchange for 1 64 plus 4 eights. In symbols:

$$\begin{array}{r} ^{1} \\ 53_8 \\ +67_8 \\ \hline 142_8 \end{array}$$

If we break up how we did this into steps, it looks like:

(1) $3+7=10=8+2=10_8+2_8$, so put down the 2 and carry the 1 eight
(2) $1+5+6=12=8+4=10_8+4_8$, so put down the 4 and carry the 1.

Subraction is the same idea:

$$\begin{array}{r} 73_8 \\ \text{-}67_8 \\ \hline 14_8 \end{array} \qquad \begin{array}{rr} 6 & 13_8 \\ \text{-}6 & 7_8 \\ \hline & 4_8 \end{array}$$

What's going on here: You can't take 7 from 3, so you convert one of the 7 eights to a 1 in the eights place giving $13_8 - 7_8$. 13_8 is one eight plus 3 ones, so taking the 7 away from the 1 eight gives 1 plus 3 ones =4. How about

$$\begin{array}{r} 103_8 \\ \text{-}67_8 \\ \hline 14_8 \end{array} \qquad \begin{array}{rr} 7 & 13_8 \\ \text{-}6 & 7_8 \\ \hline 1 & 4_8 \end{array}$$

This is the same, except instead of 7 eights, you have 1 64, which is the same as 8 eights. Taking one to add to the 3 in the ones place leaves 7 eights.
Practice: Do the base-eight arithmetic;

$$1)\ 43_8+62_8, \quad 2)\ 72_8-54_8, \quad 3)\ 106_8-57_8, \quad 4) 325_8+476_8.$$

Multiplication This is also just like base-ten, except you probably don't want to learn the multiplication tables base eight. So, you will need to convert back and forth to base ten.

$$\begin{array}{r} 46_8 \\ \times 7_8 \\ \hline 52_8 \\ 34 \\ \hline 412_8 \end{array}$$

How did we get this? Well, $7\times 6=42$ (base ten). Converting 42 to base eight is $42=52_8$, so this is the first line. $7\times 4=28=34_8$, which is the second line. To add: $2+0=2$, $5+4=9=11_8$, so put down the 1 and carry the 1.

Practice: Do the base-eight arithmetic;

$$1)\ 23_8\times 5_8, \quad 2)\ 65_8\times 7_8, \quad 3)\ 106_8\times 37_8.$$

4. Base 2

After struggling with base eight, base two will be at least 4 times as easy. In fact, you only have two digits to worry about: 0 and 1. The places are

$$2^0 = 1,\ 2^1 = 2,\ 2^2 = 4,\ 2^3 = 8,\ 2^4 = 16,\ \text{etc.}$$

For example (reading from the right):

$$11010101_2 = 1 + 0 \times 2 + 1 \times 4 + 0 \times 8 + 1 \times 16 + 0 \times 32 + 1 \times 64 + 1 \times 128 = 233.$$

Addition is really easy: In fact, here is the addition table:

+	0	1
0	0	1
1	1	10

For example:

$$\begin{array}{r} 110101001_2 \\ +\,11011011_2 \\ \hline 1010000100_2 \end{array}$$

To get you started: $1+1 = 10_2$, so put down the 0 and carry the 1. $1+0+1 = 10_2$, so put down the 0 and carry the 1, etc.

Multiplication is also very easy. Here is the base 2 multiplication table

×	0	1
0	0	0
1	0	1

For example:

$$\begin{array}{lr} & 110101001_2 \\ \times & 101_2 \\ \hline & 110101001_2 \\ & 110101001 \\ \hline & 100001001101_2 \end{array}$$

Question: What is $11010011_2 \times 1011_2$?

As you know, base 2 is perfect for computers: 0 = off, 1=on. So it is very easy for a machine to deal with base 2 arithmetic. You can also use base 2 to count to 1023 ($= 2^{10} - 1$) using only your fingers (and thumbs). A finger fully extended is a 1, a bent finger is a 0. So, extending your last two fingers on your right hand, and no others, is $11_2 = 3$. Extending your little finger on your left hand is $1000000000_2 = 1 \times 2^9 = 512$.

Practice 1. Try counting from 1 to 40 (on your fingers).

2. How do you show the following (base ten) numbers using your ten fingers:

$$32,\ 11,\ 58,\ 127,\ 129,\ 527,\ 1023$$

Converting between base two and base eight. This is very easy, because $8 = 2^3$. This means that, when converting between base 2 and base 8, you can do it in blocks, using the conversion table for 0 through 7:

Base 8	Base 2
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

Each base eight digit corresponds to a block of 3 base two digits (because the 8s place is in the second place in base 8, but is the the fourth place in base 2) For example:

$$527_8 = [101][010][111]_2 = 101010111_2$$

To go from base 2 to base eight, break up the base two number in blocks of 3 (starting at the right, and adding 0s on the left if necessary) and convert in the other direction:

$$11100101111_2 = [011][100][101][111]_2 = 3457_8$$

You can do the same converting between base four and base 2. Since $4 = 2^2$, we use blocks of 2:

$$11100101111_2 = [01][11][00][10][11][11]_2 = 130233_4$$

To go from base four to base 2, we use the same conversion table, but only need to go up to 3:

$$130233_4 = [01][11][00][10][11][11]_2 = 11100101111_2,$$

back where we started.

If you want to convert between base 4 and base 8, this simple trick doesn't work, because for example there is no eights place in base 4. One way to do it is to go via base 2:

$$32012_4 = 1110000110_2 = 1][110][000][110]_2 = 1606_8$$

or

$$7325_8 = 111011010101_2 = [11][10][11][01][01][01]_2 = 323111_4$$

Practice Convert the base two numbers to base 4 and base 8, etc.

1) 1101011_2, 2) 11011_2
3) 3102_4, 4) 2103_4
5) 741_8, 6) 6320_8

5. Base sixteen

Because each base sixteen place is the same as 4 base 2 places, base 16 is also used a lot in work with computers. Base sixteen is often called the *hexadecimal* system (hexa=6 deci=10, so 6+10=16). There is a problem with base 16: the second place is already the 16s place, so we need single digits to write all the numbers from 0 to fifteen: we need to make up new digits for the numbers ten through fifteen. The standard solution is to use letters:

base 10	base 16
10	A
11	B
12	C
13	D
14	E
15	F

So,

$$2E_{16} = 2 \times 16 + 14 = 46$$
$$C7_{16} = 12 \times 16 + 7 = 199$$

You may have seen the letters A-F (sometimes lower case) appearing when you write in your wireless card number or some other ID number from your computer; that's because these numbers are often hexadecimal.

With this one difference, base 16 arithmetic is just like base 8. Similarly, it's easy to convert between base 16 and base 2 or 4 (but to base 8, you need to go through base 4).

Practice I. Convert these numbers to base 16

1) 1101011_2, 2) 11011_2
3) 3102_4, 4) 2103_4
5) 741_8, 6) 6320_8
7) 425_{10}, 8) 1379_{10}

II. Convert these base 16 numbers to base 2 and base 10

$$1)4B_{16},\ 2)CE_{16}\ 3)B4F_{16}$$

III. Do the addition:

$$1)17_{16} + 45_{16},\ 2)C9_{16} + 98_{16}.$$

6. Other bases

There is nothing particularly special about 2, 4, 8, 16 (or 10); you can have any number as a base. The places are just the powers of the base. If the base is bigger than ten, you will need to invert other digits; usually we use the letters A, B, C, etc., as for base 16.

Note: In the Driscoll Math League sheets, they use

$$T = 10 \quad \text{(base 11, 12)}$$
$$E = 11 \quad \text{(base 12)}$$

Practice: Convert these numbers to base 10:

$$1)2104_3,\ 2)342_5,\ 3)A04_{12}, 4)6351_7$$