Answers to selected homework problems

§1.1
#10. $(x, y, z) = (-9, 5, 0)$
#20. $a = 1100$, $b = 1000$.
#34. System is inconsistent, no solution.

§1.2

#4. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ -t \\ t \end{bmatrix}$ where t is an arbitrary real number.

#18. a. No, since the third column contains two leading ones.
b. Yes
c. No, since the third row contains a leading one, but hte second row does not.
d. Yes.

#20. Four, namely $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & k \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, k an arbitrary constant.

#22. Seven, namely: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & a & b \\ 0 & 0 & 0 \end{bmatrix}$. $\begin{bmatrix} 0 & 1 & c \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & d \\ 0 & 1 & e \end{bmatrix}$, $\begin{bmatrix} 1 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Here $a, \ldots, f$ are arbitrary constants.

#30. $f(t) = 1 + 2t - t^2 - 2t^3$.

#34. We want all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = x + 3y - z = 0.$$

The endpoints of these vectors form a plane.

§1.3

2. rank is 3

4. rank is 2

6. no solution

8. Ininitely many solutions.

10. 0

12. 70

14. $\begin{bmatrix} 6 \\ 8 \end{bmatrix}$

22. By fact 1.3.4, the rref is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

24 By Fact 1.3.4, $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

26. $\text{rank}(A) = 3$ so $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Since all variables are leading, the system $A\vec{x} = \vec{c}$ cannot have infinitely many solutions, but it could have a unique solution or no solutions.

28. there must be a leading 1 in each column: $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

30. We must satisfy $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. The matrix A has rank 1 means that each column is a scalar multiple of one of the columns. So, we could take

$$\begin{bmatrix} a \\ d \\ g \end{bmatrix} = \begin{bmatrix} 2/5 \\ 0 \\ 1/5 \end{bmatrix}$$

and $\begin{bmatrix} b \\ e \\ h \end{bmatrix}$ and $\begin{bmatrix} c \\ e \\ i \end{bmatrix}$ any scalar multiple of $\begin{bmatrix} a \\ d \\ g \end{bmatrix}$. For example

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 2/5 & 0 & 0 \\ 0 & 0 & 0 \\ 1/5 & 0 & 0 \end{bmatrix}.$$

32. Similar to 30. One solution is

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} -2 & -2 & -2 \\ 3 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix}.$$

34. a) $A\vec{e}_1 = \begin{bmatrix} a \\ d \\ g \end{bmatrix}$, $A\vec{e}_2 = \begin{bmatrix} b \\ e \\ h \end{bmatrix}$, $A\vec{e}_3 = \begin{bmatrix} c \\ f \\ k \end{bmatrix}$

b) $B\vec{e}_1 = \vec{v}_1$, $B\vec{e}_2 = \vec{v}_2$, $B\vec{e}_3 = \vec{v}_3$.

36 $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$.

§2.1

2. Linear with matrix $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix}$.

6. Linear with matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

24. Rotates $90°$ counterclockwise

26. Reflects about the line $x_1 = x_2$.

28. The x_1 component is unchanged, the x_2 component is multiplied by 2.

30. Projection onto the $\vec{e}_2$ axis.

§2.2

2. This is $\begin{bmatrix} \cos(60°) & -\sin(60°) \\ \sin(60°) & \cos(60°) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

4. This is a rotation combined with a scaling: it rotates $45°$ counterclockwise and scales by a factor of $\sqrt{2}$.

6. A unit vector along L is $\begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$, so $\text{proj}_L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} \right) \cdot \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 11/9 \\ 1/9 \\ 11/9 \end{bmatrix}$.

8. A reflection along the line $x_1 = -x_2$.

10. A unit vector along L is $\begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$. So $\text{proj}_L \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \left(\begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \cdot \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 0.64x_1 + 0.48x_2 \\ 0.48x_1 + 0.36x_2 \end{bmatrix}$. Thus the matrix is $\begin{bmatrix} 0.64 & 0.48 \\ 0.48 & 0.36x_2 \end{bmatrix}$.

24. $A = [\vec{v}\ \vec{w}]$, so $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{v}$ and $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{w}$. Since A preserves length, both $\vec{v}$ and $\vec{w}$ are unit vectors. Since A preserves angles, $\vec{v}$ and $\vec{w}$ are perpendicular.

26. $k = 4$ and $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$.

§2.3.

2. rref of the matrix is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, so the matrix is not invertible.

4. the inverse is $\begin{bmatrix} 3/2 & -1 & 1/2 \\ 1/2 & 0 & -1/2 \\ -3/2 & 1 & 1/2 \end{bmatrix}$.

§2.4.

16. True: $(I_n - A)(I_n + A) = I_n^2 + A - A - A^2 = I_n - A^2$.

18. True: apply fact 2.4.8 to $B = A$.

20. Not necessarily true: $(A - B)(A + B) = A^2 - BA + AB - B^2 \neq A^2 - B^2$ if $BA \neq AB$.

22. Not necessarily true: the equation $ABA^{-1} = B$ is equivalent to $AB = BA$ (multiply by A from the right), which is not in general true.

24. True $(I_n + A)(I_n + A^{-1}) = I_n^2 + A + A^{-1} + AA^{-1} = 2I_n + A + A^{-1}$.

76. Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We want $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, which yields the equation $\begin{bmatrix} 2a - 3b & 3a + 2b \\ 2c - 3d & 3c + 2d \end{bmatrix} = \begin{bmatrix} 2a + 3c & 3d + 2b \\ 2c - 3a & -3b + 2d \end{bmatrix}$ so $a = d$ and $-b = c$. Thus B is any matrix of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.