

Review guide for exam 3

This is a list of topic headings and main ideas that I think are important, to give you a rough study guide. Your main tool should be the text and homework exercises.

§3.1. Image and kernel. You should know about:

The definition of the image and the kernel of a linear transformation.

A *linear combination* of a set of vectors.

The *span* of a set of vectors.

You should know that the image of a linear transformation with matrix A is the span of the columns of A .

§3.2 Subspaces of \mathbb{R}^n , bases and linear independence. You should know about:

The definition of a subspace of \mathbb{R}^m : a subset V of \mathbb{R}^m is a subspace if

- (1) $\vec{0}$ is in V
- (2) if v and w are in V , then $v + w$ is in V
- (3) if v is in V and k is a scalar, then kv is in V

This implies that: if v_1, \dots, v_r are in V , then every linear combination $k_1v_1 + \dots + k_rv_r$ is in V . So: If v_1, \dots, v_r are in V , then the span of v_1, \dots, v_r is contained in V .

Examples: 1. $\{0\}$ and \mathbb{R}^m are subspaces of \mathbb{R}^m .

2. If $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation, then $\ker(T)$ is a subspace of \mathbb{R}^n and $\text{im}(T)$ is a subspace of \mathbb{R}^m .

3. If v_1, \dots, v_r are vectors in \mathbb{R}^m , then $\text{span}(v_1, \dots, v_r)$ is a subspace of \mathbb{R}^m .

Linear independence: Vectors v_1, \dots, v_r in \mathbb{R}^m are *linearly independent* if the only scalars k_1, \dots, k_r for which $k_1v_1 + \dots + k_rv_r = 0$ are $k_1 = k_2 = \dots = k_r = 0$. Vectors v_1, \dots, v_r in \mathbb{R}^m are *linearly dependent* if there are scalars k_1, \dots, k_r with $k_1v_1 + \dots + k_rv_r = 0$ and with some $k_i \neq 0$.

Basis: Vectors v_1, \dots, v_r in a subspace V of \mathbb{R}^m are a *basis* for V if $V = \text{span}(v_1, \dots, v_r)$ and v_1, \dots, v_r are linearly independent.

Tests for linear independence/dependence:

1. v_1, \dots, v_r are linearly independent if $v_1 \neq 0$ and v_i is not in $\text{span}(v_1, \dots, v_{i-1})$ for $i = 2, \dots, r$.

2. v_1, \dots, v_r are linearly independent if the matrix $[v_1 \ \dots \ v_r]$ has rank r ;
 v_1, \dots, v_r are linearly dependent if the matrix $[v_1 \ \dots \ v_r]$ has rank $< r$.

v_1, \dots, v_r are linearly independent if the matrix $A := [v_1 \ \dots \ v_r]$ has $\ker A = \{\vec{0}\}$;
 v_1, \dots, v_r are linearly dependent if $\ker A \neq \{\vec{0}\}$, ie., there is a non-zero vector \vec{x} with $A\vec{x} = \vec{0}$.

3. Let V be a subspace of \mathbb{R}^m that is spanned by m vectors. Suppose you have n vectors w_1, \dots, w_n of V with $n > m$. Then w_1, \dots, w_n are linearly dependent.

Main results on bases:

- (1) Every subspace of \mathbb{R}^m has a basis (the basis of $\{0\}$ is the empty set).
- (2) If v_1, \dots, v_r and w_1, \dots, w_s are both bases of a subspace V of \mathbb{R}^m , then $r = s$.
- (3) if v_1, \dots, v_r span a subspace V , then a subset of v_1, \dots, v_r forms a basis of V .

Example: The vectors e_1, \dots, e_m of \mathbb{R}^m are a basis of \mathbb{R}^m .

§3.3. Dimension: The *dimension* of a subspace V of \mathbb{R}^m is the number of vectors in a basis for V .

Main facts on dimension:

- (1) \mathbb{R}^m has dimension m
- (2) Suppose a subspace V of \mathbb{R}^m has dimension r . Then
 - (a) If v_1, \dots, v_p are linearly independent vectors in V , then $p \leq r$.
 - (b) If v_1, \dots, v_p span V , then $p \geq r$.
 - (c) If vectors v_1, \dots, v_r span V , then v_1, \dots, v_r are automatically linearly independent.
 - (d) If vectors v_1, \dots, v_r in V are linearly independent, then v_1, \dots, v_r automatically span V .

Consequence: Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation. If any one of the following conditions is true, then all the others are true as well:

- (1) T is invertible
- (2) $\ker(T) = \{\vec{0}\}$.
- (3) $\text{im}(T) = \mathbb{R}^n$.

Finding bases: Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation with matrix A .

To find a basis for $\ker(T)$: Solve the system of equations $A\vec{x} = 0$ and get the answer in vector form:

$$\vec{x} = s_1\vec{w}_1 + \dots + s_p\vec{w}_p$$

where p is the number of free variables. Then $\vec{w}_1, \dots, \vec{w}_p$ are a basis for $\ker(T)$.

To find a basis for $\text{im}(T)$: Call a column of A a *leading column* if there is a leading 1 in the corresponding column in $\text{rref}(A)$. (There are r leading columns, where $r = \text{rank}(A)$). Let v_1, \dots, v_r be the leading columns of A . Then v_1, \dots, v_r is a basis for $\text{im}(T)$.

Conclusion: For $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation with matrix A :

1. $\dim(\ker(T)) = n - \text{rank}(A)$.
2. $\dim(\text{im}(T)) = \text{rank}(A)$.
3. $\dim(\ker(T)) + \dim(\text{im}(T)) = n$.