Review guide for exam 5

This is a list of topic headings and main ideas that I think are important, to give you a rough study guide. Your main tool should be the text and homework exercises.

§6.1, 6.2 Determinants. You should know about:

The formula for the determinant of a 2×2 matrix
How to compute a determinant of an $n \times n$ matrix by Laplace expansion.
How to compute a determinant of an $n \times n$ matrix by row reduction.

You should know that:
- The determinant of a matrix, considered as a function of the n columns, is

  a n-linear: linear in each column
  b alternating: the determinant is multiplied by -1 when you switch two columns
  c normalized: $det(I_n) = 1$.

- $\det(AB) = \det(A)\det(B)$, $\det(A^{-1}) = 1/\det(A)$, $\det(A^T) = \det(A)$
- $\det(A) \neq 0$ exactly when A is invertible. $\det(A) = 0$ exactly when A has a nonzero kernel.

§7.1. Dynamical systems. You should know that a dynamical system is given by a sequence of *states* $x(0)$, $x(1), \dots, x(m), \dots$ with the property that there is a transformation that determines the $m+1$st state $x(m+1)$ from the mth state $x(m)$. For us, the main example will be: $x(m)$ is a vector in $\mathbb{R}^n$ and there is an $n \times n$ matrix A with

$$x(m+1) = A \cdot x(n),$$

in other words $x(m) = A^m \cdot x(0)$. You should know how to compute the $x(m)$ effectively if you can find an *eigenbasis* of $\mathbb{R}^n$ for A (see below).

§7.2, 7.3, 7.4. Eigenvectors, eigenvalues and diagonalization. You should know:

- The definition of eigenvectors and eigenvalues: For an $n \times n$ matrix A, a vector $v \neq 0$ is an *eigenvector* of A with *eigenvalue* λ if $Av = \lambda v$.
- How to find eigenvalues and eigenvectors:

  (1) First find the eigenvalues by solving the *characteristic equation* $\det(A - tI_n) = 0$.
  (2) For each eigenvalue λ, find the eigenvectors with eigenvalue λ by finding a basis of $\ker(A - \lambda I_n)$.

- For an $n \times n$ matrix A, an *eigenbasis* of $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$ consisting of eigenvectors. If $v_1, \dots, v_n$ is an eigenbasis of $\mathbb{R}^n$ for A, and v_i has eigenvalue λ, let

$S = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$, and let D be the diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 \\ & & \vdots & & \\ 0 & 0 & \dots & 0 & \lambda_n \end{bmatrix}$$

Then $AS = SD$, so $S^{-1}AS = D$, $A = SDS^{-1}$. This is *diagonalizing* A. This is useful for dynamical systems, since

$$A^m = (SDS^{-1})^m = SD^mS^{-1}$$

and it is easy to compute D^m. You can diagonalize an $n \times n$ matrix A exactly when $\mathbb{R}^n$ has an basis of eigenvectors for A.

• Similar matrices: Two $n \times n$ matrices A, B are *similar* if there is an invertible matrix S with $B = S^{-1}AS$. So, A is diagonalizable exactly when A is similar to a diagonal matrix. If A and B are similar, then A and B have the same eigenvalues, in fact the same characteristic polynomial: $\det(A - tI_n) = \det(B - tI_n)$. They may not have the same eigenvectors, but if $A = T^{-1}BT$ and S is the matrix with columns the eigenvectors of A, then TS is the matrix with columns the eigenvectors of B.

• Algebraic and geometric multiplicities. If you factor the characteristic polynomial of an $n \times n$ matrix A

$$\det(A - tI_n) = (\lambda_1 - t)^{m_1} \cdot \ldots \cdot (\lambda_r - t)^{m_r}$$

then m_i is called the *algebraic multiplicity* of the eigenvalue λ_i. The dimension of the space of eigenvectors for A with eigenvalue λ_i (i.e., the dimension of $\ker(A - \lambda_i I_n)$) is called the *geometric multiplicity* of λ_i. We have

$$1 \leq \text{geom. mult.}(\lambda_i) \leq \text{algebraic mult.}(\lambda_i).$$

So, $r \leq \sum_{i=1}^r \text{geom. mult.}(\lambda_i) \leq n$, and $\sum_{i=1}^r \text{geom. mult.}(\lambda_i) = n$ exactly when $\mathbb{R}^n$ has a basis of eigenvectors for A, i.e., when A is diagonalizable. For instance, if an $n \times n$ matrix A has n different eigenvalues, then for each eigenvalue λ the space of eigenvectors with eigenvalue λ has dimenison 1, and choosing an eigenvector v_i for each λ_i, $v_1, \dots, v_n$ is an eigenbasis.

• not every matrix is diagonalizable. For instance $A = \begin{bmatrix} 5 & 1 \\ 05 & \end{bmatrix}$ has single eigenvalue $\lambda = 5$ with algebraic multiplicity 2, but only a 1-dimensional space of eigenvectors.