MTH U371-Exam 2-Oct. 4, 2007 Name____________________________

1. Write down the matrices of the following linear transformations:

a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(e_1) = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$, $T(e_2) = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

Ans. $\begin{bmatrix} 9 & -2 \\ 3 & 4 \end{bmatrix}$

b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ which rotates by 30° counterclockwise.

Ans. $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

Ans. $\begin{bmatrix} 3 & -3 \\ -2 & 4 \end{bmatrix}$

d) $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by the formula $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 - x_1 \\ x_1 + 2x_2 \\ 3x_2 - 2x_1 \end{bmatrix}$

Ans. $\begin{bmatrix} -1 & 1 \\ 1 & 2 \\ -2 & 3 \end{bmatrix}$

2. The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ rotates 45° around the y-axis and dilates by a factor of $\sqrt{2}$. The linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ is a certain shear parallel to the x-y plane. T has matrix B and S has matrix A:

$$B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}; \quad A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute the matrix of the composition $S \circ T$.

Ans. $A \cdot B = \begin{bmatrix} 4 & 0 & 2 \\ 4 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix}$

3. Here are two 3×3 matrices, one of which is invertible. For the one which is invertible, find the inverse; for the one which is not invertible, compute the reduced row echelon form and the rank. Say which is the invertible one, and which is not.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 8 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -4 & -3 \end{bmatrix}.$$

Ans. A is invertible, B is not. B has rank 2 and $\text{rref}(B) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. $A^{-1} = \begin{bmatrix} 4 & -4 & 1 \\ -5 & 7 & -2 \\ 2 & -3 & 1 \end{bmatrix}$

4. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ has the following property: for each vector $\vec{y}$ in $\mathbb{R}^n$, the equation $T(\vec{x}) = \vec{y}$ has *at least* one solution. Is T invertible? Explain.

Ans. T *is* invertible. Here is the explanation: Let A be the matrix of T. Then A is an $n \times n$ matrix, and T is invertible if A has rank $= n$. Suppose the rank of A is less than n. Then rref(A) has a 0 row. But then the system rref$(A)\vec{x} = e_n$ is inconsistent. Performing the row operations we used to row reduce A, but in reverse, we transformthe system rref$(A)\vec{x} = e_n$ to an inconsistent system $A \cdot \vec{x} = \vec{b}$. But since $T(\vec{x}) = A \cdot \vec{x}$, this means that the equation $T(\vec{x}) = \vec{b}$ has *no* solutions, which contradicts our assumption.

Therefore A has rank $= n$ and T is invertible.

5. Here are three matrices:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}; \quad C = \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}.$$

Compute all the matrix products you can form by multiplying two of the matrices (or a matrix with itself).

Ans. There are five products that make sense: $AB = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$; $BA = \begin{bmatrix} 1 & 0 & 1 \\ 3 & -2 & 1 \\ 1 & -1 & 0 \end{bmatrix}$;

$CA = \begin{bmatrix} -1 & 1 & 0 \\ 3 & 0 & 3 \end{bmatrix}$; $BC = \begin{bmatrix} 0 & -1 \\ 6 & -1 \\ 3 & 0 \end{bmatrix}$; $C^2 = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$.