

1. (20 points) Let $T : \mathbb{R}^5 \rightarrow \mathbb{R}^4$ with matrix $A = \begin{bmatrix} 1 & 2 & 5 & 1 & -2 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & -1 & -1 & 3 & -5 \end{bmatrix}$. Find a

basis for $\ker(T)$ and a basis for $\text{im}(T)$.

Ans. First row reduce A :

$$\begin{bmatrix} 1 & 2 & 5 & 1 & -2 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & -1 & -1 & 3 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 3 & -4 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The leading 1s in $\text{rref}(A)$ are in column 1, 2 and 4, so these columns in A give a basis for $\text{im}(A)$:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}$$

are a basis for $\text{im}(A)$.

For $\ker(A)$: solve $A\vec{x} = \vec{0}$ in vector form.

$$x_1 = -3x_3 - 2x_5$$

$$x_2 = -x_3 + x_5$$

$$x_4 = 2x_5$$

Set $x_3 = s$, $x_5 = t$, and the solution in vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$$

and $\ker(A)$ has basis

$$\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}.$$

2. (10 points) Are the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 7 \\ 8 \\ 0 \\ 9 \end{bmatrix}$ linearly independent? Explain

Ans. Yes they are linearly independent. Here are two explanations:

i. Call the vectors v_1, v_2, v_3 . v_1 is not zero. v_2 has a 3 in the 2nd position, while v_1 has a 0 there, so v_2 is not a scalar multiple of v_1 . Both v_1 and v_2 have a 0 in the 4th position, while v_3 has a 9 there, so v_3 is not in the span of v_1, v_2 . This

shows that v_1, v_2, v_3 are linearly independent.

ii. Let $A = [v_1, v_2, v_3]$. If we exchange rows 3 and 4, the matrix is now upper triangular with non-zero entries on the diagonal, so when we row reduce further, we get 3 leading 1s. Thus A has rank 3, so v_1, v_2, v_3 are linearly independent.

3. (20 points) A linear transformation $T : \mathbb{R}^6 \rightarrow \mathbb{R}^5$ has matrix A . You find

that the reduced row echelon form of A is
$$\begin{bmatrix} 1 & 0 & 5 & 1 & 0 & -3 \\ 0 & 1 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a. Find a basis for $\ker(T)$.

Ans. This is just like (1): Write the solution in vector form:

$$x_1 = -5x_3 - x_4 + 3x_6$$

$$x_2 = -3x_3 - 2x_4$$

$$x_5 = -2x_6$$

With $s_1 = x_3, s_2 = x_4, s_3 = x_6$, this gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = s_1 \begin{bmatrix} -5 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

so

$$\begin{bmatrix} -5 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

is a basis for $\ker(T)$.

b. What is the dimension of $\text{im}(T)$? Explain.

Ans. $\dim(\text{im}(T)) + \dim(\ker(T)) = \#$ of columns of $A = 6$. From (a) $\dim(\ker(T)) = 3$, so $\dim(\text{im}(T)) = 3$ as well.

4. (15 points) For what values of k does the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & k \\ 4 & 9 & k^2 \end{bmatrix}$ have $\dim(\ker(A)) = 1$?

Ans. $\dim(\ker(A)) + \dim(\text{im}(A)) = \#$ of cols of $A = 3$, so $\dim(\ker(A)) = 1$ means

$\text{rank}(A) = 2$. If you partially row reduce A , you get

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & k \\ 4 & 9 & k^2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & k-2 \\ 0 & 5 & k^2-4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & k-2 \\ 0 & 0 & k^2-5k+6 \end{bmatrix}$$

so A has rank 2 when $0 = k^2 - 5k + 6 = (k-2)(k-3)$, i.e., for $k = 2, 3$.

5. (20 points) Find a basis of the subspace of \mathbb{R}^4 spanned by the vectors

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ -2 \\ -4 \end{bmatrix}.$$

Ans. Let $A = \begin{bmatrix} 0 & 3 & 3 & -4 \\ 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & -4 \end{bmatrix}$ (Partially) row reduce A :

$$\begin{bmatrix} 0 & 3 & 3 & -4 \\ 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 3 & 3 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 3 & 3 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

from which it is clear that $\text{rref}(A)$ will have a leading 1 in columns 1,2,3. Thus the first three columns of A :

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \\ 2 \end{bmatrix}.$$

are a basis of the subspace spanned by the four given vectors.

6. (15 points) You have a linear transformation $T : \mathbb{R}^{15} \rightarrow \mathbb{R}^9$. You know that the vector $(1, 2, 3, 4, 5, 6, 7, 8, 9)$ is *not* in the image of T . Knowing this, but nothing else, about T , what can you conclude about the dimension of the kernel of T ? Explain.

Ans. We have

$$\dim(\text{im}(T)) + \dim(\text{ker}(T)) = 15$$

Since there is a vector in \mathbb{R}^9 which is not in the image of T , we know that

$$0 \leq \dim(\text{im}(T)) \leq 8$$

Thus $7 \leq \dim(\text{ker}(T)) \leq 15$.