RANDOM WALKS ON SYMMETRIC SPACES AND SINGULAR SPECTRUM OF MATRIX PRODUCT

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Abstract. Singular spectrum $\sigma(A)$ of complex matrix A is usual spectrum

$$\lambda(A\cdot A^*)\ :\ \lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$$

of nonnegative Hermitian matrix $A\cdot A^*$.

Theorem 1. *The following conditions are equivalent*

(1) *There exist matrices $A_i\in\mathrm{GL}(n,\mathbb{C})$ with given singular spectra*

$$\sigma_i=\sigma(A_i)\ \textit{and}\ \sigma=\sigma(A_1A_2\cdots A_m).$$

(2) *There exist Hermitian matrices H_i with spectra*

$$\lambda(H_i)=\log\sigma_i\ \textit{and}\ \lambda(H_1+H_2+\cdots+H_m)=\log\sigma.$$

Remark. It is explicitly known a system of inequalities necessary and sufficient for existence of Hermitian matrices H_i with given spectra $\lambda_i=\lambda(H_i)$ and $\lambda=\lambda(H_1+H_2+\cdots+H_m)$. This inequalities will be a subject of another talk.

We derive theorem 1 from study of *random walks* (whatever it is) on three symmetric spaces

1. Compact simply connected Lie group G
2. Its Lie algebra $\mathcal{L}=\mathcal{L}(G)$
3. $X_G=G_{\mathbb{C}}/G$

of positive, zero and negative curvature. For unitary group $G=\mathrm{SU}(n)$ Lie algebra $\mathcal{L}$ is the space of (anti) Hermitian traceless matrices, while $X=X_G$ is space of positive unimodular Hermitian matrices. The random walk on X we are speaking about is given by distribution of product $A_1A_2\cdots A_m$ of random matrices with given singular spectra $\sigma_i=\sigma(A_i)$. In general the multipliers A_i should be uniformly distributed in given orbits $\mathcal{O}_i\subset X_G$ of group G. Random walks on G and $\mathcal{L}$ are defined in a similar way using adjoint orbits. For Lie algebra of $\mathrm{SU}(n)$ it is given by distribution of sum $H_1+H_2+\cdots+H_m$ of random Hermitian matrices with given spectra $\lambda_i=\lambda(H_i)$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Theorem 2. *The densities of the random walks on G, $\mathcal{L}$ and X are related by the following equations*

$$\begin{aligned} p_{\mathcal{L}}(\gamma) &= p_X(\exp i\gamma)\prod_{k=0}^{m}\prod_{\alpha>0}\frac{\sinh(\gamma_k,\alpha)}{(\gamma_k,\alpha)} \\ &= p_G(\exp\gamma)\prod_{k=0}^{m}\prod_{\alpha>0}\frac{\sin(\gamma_k,\alpha)}{(\gamma_k,\alpha)} \end{aligned}$$

where the last equality holds for sufficiently small γ_k = *representatives of the adjoint orbits* $\mathcal{O}_k$ *in positive chamber. Here* $\gamma \in \mathcal{L}$ *and* α *are positive roots.*

Since $p_{\mathcal{L}}$ and p_X differ by a nonvanishing multiplier, then the probability distributions have in essential the same supports: $\operatorname{supp}(p_X) = \exp(\operatorname{supp}(p_{\mathcal{L}}))$. For unitary group this implies theorem 1.