Matrix, Matrices

As is common in multidimensional calculus, points in space are represented by vectors. For example

$$\boldsymbol{B} \ = \ \begin{bmatrix} 5 \\ 0 \\ -9 \end{bmatrix}$$

is the three-dimensional column vector that represents the point (5, 0, -9) in three$-$dimensional space.
We can also write our vectors in "row" form:

$$[3 \quad 2 \quad -9]$$

Two vectors are equal if a. they are both columns, or both rows,
and b. they have the same dimensions, that is, the same number of entries
and c. the entries are each equal to the corresponding entry of the other

The basic operations on vectors are multiplication by a "scalar" (that is a real number)

$$3\boldsymbol{B} \ = \ \begin{bmatrix} 15 \\ 0 \\ -27 \end{bmatrix}$$

and addition

$$\begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix} \ + \ \begin{bmatrix} -4 \\ -3 \\ 4 \end{bmatrix} \ = \ \begin{bmatrix} 1 \\ -5 \\ 6 \end{bmatrix}$$

Two vectors can be added together if they are the same size and type and *only* if they are the same size and type.

We define a matrix to be a recatangular array of numbers. A matrix can be thought of as a horizontal list of column vectors of the same size, or alternatively as a vertical list of row vectors of the same size. For example

$$\boldsymbol{C} \;=\; \begin{bmatrix} 2 & 1 & -1 & 5 \\ 4 & -2 & 0 & 0 \\ 6 & -7 & 1 & -9 \end{bmatrix}$$

is a 3 × 4 matrix (read “three by four matrix"). It has three rows and four columns and can be thought of either as 4 column vectors of length 3, or 3 row vectors of length 4.

Two matrices are equal if a. they have the same dimensions, that is, the same number of rows and the same number of columns

and b. the entries are each equal to the corresponding entry of the other

The basic operations on matrices are multiplication by a scalar

$$3\boldsymbol{C} \;=\; \begin{bmatrix} 6 & 3 & -3 & 15 \\ 12 & -6 & 0 & 0 \\ 18 & -21 & 3 & -27 \end{bmatrix}$$

and addition

$$\begin{bmatrix} 2 & 1 \\ -3 & 2 \\ 0 & 4 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 6 \\ 4 & -2 \\ 4 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 1 & 0 \\ 4 & 3 \\ 2 & 0 \end{bmatrix}$$

These are 4 × 2 matrices. Two matrices can be added together if they have the same dimensions and only if they have the same dimensions.

How do we *multiply* two matrices? We first answer the question for two special matrices; namely, the product of a 1 $\times$ n matrix, which is a row vector, and an n $\times$ 1 matrix, which is a column vector. It is done as follows

$$\begin{bmatrix} 4 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = [(4 \cdot 3) + (1 \cdot 1) + (3 \cdot 0)] = [\,13\,]$$

This is sometimes called the *dot product* of two vectors. To extend the definition to the product of a matrix and a column vector, just take the product of each row of the matrix with the column vector and stack the results in a new column vector:

$$\begin{bmatrix} 4 & 1 & 3 \\ 2 & 6 & 8 \\ 1 & 0 & 9 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (4 \cdot 3) + (1 \cdot 1) + (3 \cdot 0) \\ (2 \cdot 3) + (6 \cdot 1) + (8 \cdot 0) \\ (1 \cdot 3) + (0 \cdot 1) + (9 \cdot 0) \\ (2 \cdot 3) + (2 \cdot 1) + (1 \cdot 0) \end{bmatrix} = \begin{bmatrix} 13 \\ 12 \\ 3 \\ 8 \end{bmatrix}$$

Note that the number of columns of the matrix must equal the dimension of the vector being multiplied.

This definition can be used to represent a system of linear equations

$$\begin{aligned} 2u + \ \ v - \ w &= 5 \\ 4u - 2v \qquad &= 0 \\ 6u - 7v + \ w &= -9 \end{aligned}$$

as a matrix multiplying an unknown vector equalling a known one as follows:

$$\begin{bmatrix} 2 & 1 & -1 \\ 4 & -2 & 0 \\ 6 & -7 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -9 \end{bmatrix}$$

If we multiply out the left side we get an equation expressing the equality of two vectors.

$$\begin{bmatrix} 2u + \ v - \ w \\ 4u - 2v \qquad \\ 6u - 7v + \ w \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -9 \end{bmatrix}$$

If we equate each of the three components (or entries) of these vectors we get the original equations.

This is an equation of the form $\boldsymbol{Ax} = \boldsymbol{b}$ where the known matrix $\boldsymbol{A}$, called the coefficient matrix of the system, multiplies the unknown vector $\boldsymbol{x}$ and equals the known vector $\boldsymbol{b}$. The problem is to find the vector $\boldsymbol{x}$. Note that the solution vector

$$\boldsymbol{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

satisfies the equation

$$\begin{bmatrix} 2 & 1 & -1 \\ 4 & -2 & 0 \\ 6 & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -9 \end{bmatrix}$$

This shows that our definition of a matrix times a vector has some practical value and is not just some more mathematcal generalised abstract nonsense. We’ll see, soon enough, that matrices can also represent a system of *differential* equations.

Finally, to multiply two matrices just multiply the left matrix times each column of the right matrix and line up the resulting two vectors in a new matrix. For example

$$\begin{bmatrix} 4 & 1 & 3 \\ 2 & 6 & 8 \\ 1 & 0 & 9 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 23 \\ 12 & 18 \\ 3 & 14 \\ 8 & 11 \end{bmatrix}$$

Note again that the number of *columns* of the left factor must equal the number of *rows* of the right factor for this to make sense.

In the example above we multiplied a 4×3 matrix times a 3×2 matrix and obtained a 4×2 matrix. In general, if $\boldsymbol{A}$ is $m \times n$ and $\boldsymbol{B}$ is $n \times p$, then $\boldsymbol{AB}$ is $m \times p$.

Matrix multiplication satisfies the associative law, $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$, and the two distributive laws, $\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{AB}+\boldsymbol{AC}$ and $(\boldsymbol{B}+\boldsymbol{C})\boldsymbol{D} = \boldsymbol{BD}+\boldsymbol{CD}$. It does not, however, satisfy the commutative law. That is, in general $\boldsymbol{EF} \neq \boldsymbol{FE}$. For example

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

In fact, for many pairs of matrices, I can multiply $\boldsymbol{EF}$ but not $\boldsymbol{FE}$. (See Exercise 2.)

For every n there is a special $n \times n$ matrix, which we denote $\boldsymbol{I}$, with 1's down its diagonal and zero's everywhere else. For example, in the 3×3 case

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is easy to see that for any 3×3 matrix $\boldsymbol{A}$ we have $\boldsymbol{IA} = \boldsymbol{AI} = \boldsymbol{A}$. For this reason $\boldsymbol{I}$ is called the <u>identity</u> <u>matrix</u>.

As a final bit of notation, we write the general matrix $\boldsymbol{A}$ with m rows and n columns as

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

where a_{ij} denotes the entry in the ith row and the jth column.

Exercises

1. Compute the following:

(a) $2\begin{bmatrix} 5 & 7 & -1 \\ 4 & -2 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 6 & 2 \\ 7 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 3 & 6 \\ 5 & -7 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 4 & -5 \end{bmatrix} \begin{bmatrix} 4 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

(f) $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

(g) $\begin{bmatrix} 2 & -1 & 3 \\ 5 & 0 & 7 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 1 \\ -2 & 3 \end{bmatrix}$

(h) $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 5 & 0 & 7 \\ 0 & -1 & 0 \end{bmatrix}$

(i) $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. Which of the expressions $2\boldsymbol{A}$, $\boldsymbol{A} + \boldsymbol{B}$, $\boldsymbol{AB}$, and $\boldsymbol{BA}$ makes sense for the two matrices below? Which do not?

$$\boldsymbol{A} = \begin{bmatrix} 5 & 7 & -1 \\ 4 & -2 & 0 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

3. Give 3×3 matrices that are examples of the following.

(a) diagonal matrix: $a_{ij} = 0$ for all $i \neq j$.
(b) symmetric matrix: $a_{ij} = a_{ji}$ for all i and j.
(c) upper triangular matrix: $a_{ij} = 0$ for all $i > j$.

(d) stochastic matrix: $0 \leq a_{ij} \leq 1$ for all i and j, and $\sum_{i=1}^{n} a_{ij} = 1$ for all j.

(e) predator-prey matrix: $a_{ij}\, a_{ji} < 0$ for all $i \neq j$, and $a_{ii} \leq 0$ for all i.

4. Show with a 3×3 example that the product of two upper triangular matrices is upper triangular.

5. Show with a 3×3 example that the product of two nonzero matrices can be $\boldsymbol{Z}$, the zero matrix (the matrix consisting of nothing but zeros).

6. For any matrix $\boldsymbol{A}$, we define its transpose $\boldsymbol{A}^{\mathrm{T}}$ to be the matrix whose columns are the corresponding rows of $\boldsymbol{A}$.

(a) What is the transpose of $\begin{bmatrix} 2 & -1 & 3 \\ 5 & 0 & 7 \\ 0 & -1 & 0 \end{bmatrix}$?

(b) If a matrix satisfies $\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}$, then what kind of matrix is it? (See problem 3 above.)

(c) The formula $(\boldsymbol{AB})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$ holds as long as the product $\boldsymbol{AB}$ makes sense. Illustrate this with a 3×3 example.

(d) Show that for any matrix $\boldsymbol{C}$, the matrices $\boldsymbol{CC}^{\mathrm{T}}$ and $\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C}$ are both symmetric. (Use (b) and (c) above.)

7. Verify $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix} = 3 \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} - 5 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Some Answers

1. (a) $\begin{bmatrix} 10 & 14 & -2 \\ 8 & -4 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & 0 \\ 10 & 7 \\ 6 & -5 \end{bmatrix}$ (c) $\begin{bmatrix} 17 \\ 4 \\ -7 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 14 & -8 \end{bmatrix}$

(e) $\begin{bmatrix} 32 \end{bmatrix}$ (f) $\begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix}$ (g) $\begin{bmatrix} -8 & 10 \\ -14 & 26 \\ -2 & -1 \end{bmatrix}$ (h) $\begin{bmatrix} 8 & -3 & 12 \\ 5 & 0 & 7 \\ -6 & -3 & -8 \end{bmatrix}$

(i) $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$

6. (a) $\begin{bmatrix} 2 & 5 & 0 \\ -1 & 0 & -1 \\ 3 & 7 & 0 \end{bmatrix}$

More
Matrix Homework

1) Let $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 4 & -2 \\ -3 & a \end{bmatrix}$

compute

a.AB
b.BA
c. AI
d. $B + I$
e. $B - I$
f. $B - 2I$
g. $B - 5I$
h. $B - rI$
i. $3B + 2A + C$
j. $A(B + I)$

2) If B is any square matrix describe in words what the following operations do:

a. $B + I$
b. $B - I$
c. $B - 5I$
d. $B - rI$

3) Let $A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 & 1 \\ -1 & 0 & -2 \\ 0 & 3 & 0 \end{bmatrix}$

compute

a. $A + B$
b. $A - B$
c. $2A + B$
d. $A - I$
e. AB

Probable answers:

1) a.$\begin{bmatrix} 5 & 8 \\ 5 & 6 \end{bmatrix}$ b.$\begin{bmatrix} 0 & 5 \\ 2 & 11 \end{bmatrix}$ c.$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ d.$\begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$ e.$\begin{bmatrix} 0 & 2 \\ 3 & 3 \end{bmatrix}$ f.$\begin{bmatrix} -1 & 2 \\ 3 & 2 \end{bmatrix}$ g.$\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$ h.$\begin{bmatrix} 2-r & 2 \\ 3 & 5-r \end{bmatrix}$ i.$\begin{bmatrix} 11 & 6 \\ 4 & 16+a \end{bmatrix}$ j.$\begin{bmatrix} 7 & 9 \\ 4 & 8 \end{bmatrix}$

2) a. Add 1 to the diagonal. b.Subtract 1 from the diagonal. c.Subtract 5 from the diagonal. d.Subtract r from the diagonal.

3) a.$\begin{bmatrix} 2 & 2 & 2 \\ -2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ b.$\begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & 5 \\ 0 & -5 & 1 \end{bmatrix}$ c.$\begin{bmatrix} 4 & 2 & 3 \\ -3 & 2 & 4 \\ 0 & -1 & 2 \end{bmatrix}$ d.$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 3 \\ 0 & -2 & 0 \end{bmatrix}$ e.$\begin{bmatrix} 0 & 7 & 2 \\ -1 & 7 & -3 \\ 2 & 3 & 4 \end{bmatrix}$