VECTOR BUNDLES AND HERMITIAN OPERATORS

ALEXANDER A. KLYACHKO

ABSTRACT. We'll deal with three apparently disjoint problems:
1. The spectrum of a sum of Hermitian operators,
2. Components of tensor product of irreducible representations of the group $\mathrm{GL}_n(\mathbb{C})$,
3. Structure of the moduli space of stable bundles on the projective plane $\mathbb{P}^2$.

1. HERMITIAN OPERATORS

We begin with the first subject. Let $A : E \to E$ be a Hermitian operator in a unitary space E of finite dimension n and let

$$\lambda(A) \ : \ \lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$$

be its spectrum. The problem is to find all restrictions on spectra $\lambda(A), \lambda(B)$ and $\lambda(A+B)$. First of all we have *trace identity*

$$\sum_i \lambda_i(A+B) = \sum_i \lambda_i(A) + \sum_i \lambda_i(B)$$

and a number of classical inequalities, all of the form

$$\text{(IJK)} \qquad \sum_{k \in K} \lambda_k(A+B) \leq \sum_{i \in I} \lambda_i(A) + \sum_{j \in J} \lambda_j(B)$$

for some triple of subsets $I, J, K \subset \{1, 2, \ldots, n\}$ of the same cardinality.

Let us fix a decomposition $n = p + q$. We have a bijection between subsets $I \subset \{1, 2, \ldots, n\}$ of cardinality $p = |I|$ and Young diagrams $\sigma = \sigma_I$ in a rectangular box of dimension p (North) by q (East) given as follows. Let $\Gamma = \Gamma_I$ be a polygonal line with unit edges that runs from the South-West corner of the box to the East-North corner with the i-th edge running to the North for $i \in I$ and to the East otherwise. The line $\Gamma = \Gamma_I$ cuts out from the box a Young diagram $\sigma = \sigma_I \subset p \times q$ situated in its North-West angle. The diagram σ_I in the usual way corresponds to a Schubert cycle s_I in a Chow ring of the Grassmannian.

Theorem 1.1. *Consider a triple of subsets $I, J, K \subset \{1, 2, \ldots, n\}$ such that the Schubert cycle s_K is a component of $s_I \cdot s_J$. Then*

i) *The inequality* (IJK) *holds.*
ii) *In union with the trace identity, this inequalities form a complete set of restrictions on spectra of A, B and $A + B$.*

2. TENSOR PRODUCTS

Let us consider an integer spectrum

$$\alpha \ : \ a_1 \geq a_2 \geq \cdots \geq a_n, \qquad a_i \in \mathbb{Z},$$

and associate with it the following dominant weight of general linear group $GL_n(\mathbb{C})$

$$\omega^\alpha : \operatorname{diag}(x_1, x_2, \dots , x_n) \mapsto x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

The weight ω^α in the usual way corresponds to an irreducible representation $V(\omega^\alpha)$ of the group $\mathrm{GL}_n(\mathbb{C})$ with highest weight ω^α.

We are iterested in the problem: which irreducible representations $V(\omega^\gamma)$ are components of tensor product $V(\omega^\alpha) \otimes V(\omega^\beta)$? The result is that the inequalities (IJK) of Theorem 1.1 answer this question too. More precisely,

Theorem 2.1. *The irreducible representation $V(\omega^{N\gamma})$ is a component of tensor product $V(\omega^{N\alpha}) \otimes V(\omega^{N\beta})$ for some positive N if and only if α, β and γ are spectra of Hermitian operators A, B and $C = A + B$.*

3. Vector bundles

It will be expalined in the lecture that both of the previouse theorems follows from existence of Hermite–Einstein metric on stable toric vector bundles on $\mathbb{P}^2$.