Embedding 3-manifolds in 4-space

The question of which 3-manifolds embed in 4-space seems very natural, but has not been much studied, except for the case of homology 3-spheres. One of the difficulties is that there is (as yet) no systematic listing of all closed 3-manifolds. A more subtle issue is that the answer depends on the meaning of "embedding". Freedman showed that *every* homology 3-sphere embeds topologically in 4-space, whereas the question of smooth embeddings remains largely open.

Freedman's approach may be adapted to other 3-manifolds which are homologically like 3-manifolds with torsion-free solvable groups. (The idea is very simple, although it needs machinery for its justification.) Manifolds of the latter type are Seifert fibred, and the class of Seifert fibred 3-manifolds does have a useful parametrization in terms of Seifert data. Thus it is reasonable to hope for an answer to our question for such manifolds.

In joint work with John Crisp, we used the $Z/2Z$-Index Theorem to show that there are exactly twelve closed 3-manifolds with virtually solvable fundamental group that embed smoothly, and one more that embeds topologically. We also found strong constraints on the Euler invariant of Seifert manifolds over nonorientable bases which embed (generalizing work of Massey).

Recent work suggests a complete criterion for Seifert fibred manifolds with orientable base, no cone points of even order and Euler invariant 0. This criterion is sufficient (by direct construction), and another application of the G-Index Theorem shows it to be necessary if also the base is a marked 2-sphere $S^2(\alpha_1, \ldots, \alpha_r)$.

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