Von Neumann Algebras and Applications
MTH G354, Fall 2006
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Class meetings: Monday and Wednesday 7:30 - 9:00 p.m. 509 Lake Hall

Textbook: No textbook is needed. The necessary materials from a book in preparation will be provided.

Course description. Von Neumann algebras and von Neumann dimensions are among the most beautiful and useful inventions of mathematics of the 20th century.

Let us recall that the dimension of a finite-dimensional linear space in Linear Algebra can be $0, 1, 2, \ldots$. It is used to compare "size" of different subspaces, count the number of linearly independent solutions of systems of linear equations etc.

In 1929 J. von Neumann initiated a new direction in Functional Analysis, which was developed in his joint papers with F.J.Murray in the mid 1930s. One of the important features of the theory is that it is possible to measure some infinite-dimensional subspaces in a Hilbert space, assigning to them dimensions which are arbitrary real (and not necessarily integer) numbers. For example the von Neumann dimension of a subspace can be equal to $\sqrt{2}$, $\pi/4$ etc. Besides these dimensions have all properties of the usual dimension (which can only take values $0, 1, 2, \ldots$ and $+\infty$). The only difference is that a von Neumann dimension does not measure *all* (closed) subspaces, but only some special class of them: the ones which are affiliated with a special algebra of operators (i.e. such that the corresponding orthogonal projection belongs to the algebra). The most appropriate algebras of operators were the main object of study in the papers quoted above, and they are now called the von Neumann algebras or W^*-algebras (though in some terminology these two terms refer to slightly different objects).

Von Neumann dimensions and von Neumann algebras proved to be quite useful in Analysis (including Complex Analysis), Geometry, Topology and Statistical Mechanics, where appropriate infinite-dimensional spaces arise as spectral subspaces of periodic or almost-periodic differential operators (which usually have continuous spectrum), the spaces of harmonic forms on covering manifolds etc. Recently an extensive use of these ideas was made in the theory of L^2-invariants of manifolds and more general CW-complexes.

The applications to be discussed will include index theories of non-Fredholm operators, L^2 Betti numbers and other L^2 invariants of manifolds, spectral properties of Laplacians and Schrödinger operators on non-compact manifolds.

The necessary prerequisites are a subset of the most standard material from Analysis, Algebra and Topology, some Functional Analysis and elementary measure theory. I will try to make the course as self-contained as possible. In particular, many facts from Functional Analysis, which are not entirely standard, will be provided with detailed proofs or sketches of the proof. The same holds for pseudo-differential operators and Sobolev spaces.

The grade will be based on home assignments.