Probability, Quantum Field Theory, and Geometry

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The classical project of constructive quantum field theory, as formulated by Wightman, Nelson, Glimm, and Jaffe, was to develop the mathematical foundations of quantum field theory by constructing appropriate non-Gaussian probability measures on infinite-dimensional spaces. These measures are the mathematical versions of Feynman's path integral approach to quantum field theory. The technical problems involved in these constructions arise from the fact that these measures are supported on spaces of distributions, and therefore the existence of nonlinear perturbations is very difficult to establish. In physics these difficulties are expressed as the appearance of singularities in the expectations of random variables in these measures, and the physicists' method for extracting information from these singular computations is known as **renormalization.** This classical development of constructive quantum field theory was a modernized version of Hilbert's Sixth Problem of developing the mathematical foundations of physics.

As this program was approaching its goals in the 1980's, the scene was revolutionized by Witten's work on the application of ideas from quantum field theory to geometry and topology. In Witten's work the intuitive path integrals developed by Feynman were generalized to vastly different settings, and calculational techniques appropriate to physical models were developed to yield precise geometrical and topological conjectures utterly unexpected by mathematicians. Entire fields of geometry and topology were developed to understand these calculations without reference to the underlying path-integral techniques. Nevertheless the mathematics underlying these generalizations of Feynman path integrals was never developed into a geometric version of constructive field theory, and geometers and topologists remain entirely dependent on the physicists' intuition and in their interest in developing new conjectures for mathematicians to prove. There is clearly an important piece of mathematics which is entirely missing and which the physicists have guessed at without being able to articulate a consistent theory. In essence the problem remaining is one of developing the mathematical foundations of much of modern mathematics.

In this lecture series we will describe the basics of quantum field theory, both in its origins in physical (or almost physical) models, and in the models used in geometry and topology. We will give some idea of the methods used in classical constructive quantum field theory, and then show how techniques developed in the past two years, which combine insight from geometry with ideas from constructive quantum field theory, yield non-Gaussian probability measures corresponding to the supersymmetric quantum field theories arising in geometry and topology. We will then give an idea of the mathematics that may be expected to result from the existence of these probability measures, in the cases considered by the physicists and beyond.

We will assume a basic background in differential geometry and analysis on the level of a first-year graduate course. Knowledge of probability is helpful but we will cover the necessary results.