Northeastern University
Mathematics Department

Qualifying Exam, General Algebra
September 2006

(1) Suppose

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1 & \xrightarrow{g} & V & \xrightarrow{h} & V_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_2} & & \\ 0 & \longrightarrow & V_1 & \xrightarrow{g} & V & \xrightarrow{h} & V_2 & \longrightarrow & 0 \end{array}$$

is a commutative diagram in the category of vector spaces and that the rows are exact. Suppose also that both f_1 and f_2 are the zero maps. Is f also the zero map? If it is, give a proof. If not, give a counterexample.

(2) Let V be a vector space over a field K. Let $e_V : V \to V^{**}$, where $(-)^* = \mathrm{Hom}_K(-, K)$, be the canonical evaluation map: $e_V(v)$ is the map that evaluates each functional on V at $v \in V$. By the adjointness property of the Hom and tensor-product functors, we have a natural isomorphism

$$\mathrm{Hom}_K(V \otimes V^*, K) \to \mathrm{Hom}_K(V, V^{**}).$$

Give an explicit description of the linear map $f : V \otimes V^* \to K$ corresponding to $e_V \in \mathrm{Hom}_K(V, V^{**})$ under this isomorphism.

(3) Find the characteristic polynomial of the following matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & a_n \\ -1 & 0 & 0 & \dots & 0 & 0 & a_{n-1} \\ 0 & -1 & 0 & \dots & 0 & 0 & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 & 0 & a_2 \\ 0 & 0 & 0 & \dots & 0 & -1 & a_1 \end{pmatrix}$$

(4) The polynomial $f(x) = x(x-2)^2(x-3)$ vanishes on a 4×4 matrix A. Describe all possible Jordan normal forms of A.

(5) Let A be a square matrix of size n. The Cayley-Hamilton theorem asserts that $P_A(A) = 0$, where $P_A(t) = \det(A - tI_n)$ is the characteristic polynomial of A. Comment on the following short argument:

$$P_A(A) = \det(A - AI_n) = \det(0_n) = 0.$$

Is this argument correct or incorrect? If it is correct, is it really shorter than the proof of the Cayley - Hamilton theorem that you know? If it is incorrect, explain why.

(6) Describe all abelian non-cyclic groups of order 1200 (up to isomorphism).

(7) Describe $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z})$, the set of all group homomorphisms from the group $\mathbb{Z}/3\mathbb{Z}$ to the group $\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$.

(8) Consider S_5, the group of permutations of 5 elements. Prove that its 3-Sylow subgroups are not normal.

(9) Let p and q be distinct primes. Show that any action of a p-group on a set of q elements has a fixed point.

(10) Anything that you studied, or you really like, but we didn't ask? Please, state the problem AND solve it.