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QUALIFYING EXAM IN ALGEBRA. January 2008.

1. We work over the field of real numbers. Reduce the quadratic form

$$q(x_1, \dots, x_4) = \sum_{1 \le i < j \le 4} x_i x_j$$

to the diagonal form and calculate its signature.

2. Use universality property to prove that for a finite dimensional vector space V there is a canonical isomorphism

$$\phi : \bigwedge^i (V^*) \to (\bigwedge^i V)^*.$$

3. Let V be a vector space of dimension 3 with the basis $\{e_1, e_2, e_3\}$ and let W be a vector space of dimension 2 with the basis $\{f_1, f_2\}$. Which of the following tensors are decomposable (i.e. of the form $v \otimes w$ with $v \in V$, $w \in W$).

a) $s = e_1 \otimes f_1 + e_2 \otimes f_2$,

b) $t = e_1 \otimes f_2 + e_1 \otimes f_3 + e_2 \otimes f_2 + e_2 \otimes f_3$,

c) $u = e_1 \otimes f_2 + e_1 \otimes f_3 + e_2 \otimes f_2$.

What is the condition for a tensor

$$v = \sum_{i=1}^{3} \sum_{j=1}^{2} a_{i,j} e_i \otimes f_j$$

from $V \otimes W$ to be decomposable.

4. Let $f : K^3 \to K^3$ be a map in Jordan canonical form having a matrix

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Find Jordan canonical form of $f \otimes f$.

5. Let $\mathbf{R}^4$ be the Euclidean space with the scalar product givern by the usual dot product. Let V be a subspace spanned by $(1, 1, 0, 0)$, $(0,1,1,0)$, $(0, 0, 1, 1)$. Apply the Gram-Schmidt process to find the orthonormal basis of V. Find the orthogonal complement of V.

6. List all isomorphism classes of Abelian groups of order 400.

7. Describe the Abelian group $Hom_{Ab}(\mathbf{Z}/5\mathbf{Z}, \mathbf{Z} \oplus \mathbf{Z}/10\mathbf{Z})$.

8. Describe all conjugacy classes of 6×6 matrices A which satisfy the equation $A^3(A - 1)^3 = 0$.

9. Give the examples of groups of the following type:

a) An infinite non-Abelian group,

b) An infinite torsion Abelian group,

c) A homomorphism of groups which is neither injective nor surjective,

d) A subgroup which is not normal,

10. Let G be an Abelian group occurring in an exact sequence

$$0 \to \mathbf{Z}/4\mathbf{Z} \to G \to \mathbf{Z}/8\mathbf{Z} \to 0.$$

What can you say about G ?