

Northeastern University
Mathematics Department

Qualifying Exam, Algebra Concentration
February 2007

- (1) Decide which of the following functors are exact functors $Mod\mathbb{Z} \rightarrow Mod\mathbb{Z}$:
 $F_1 = Hom_{\mathbb{Z}}(_, \mathbb{Z})$, $F_2 = Hom_{\mathbb{Z}}(\mathbb{Z}, _)$, $F_3 = Hom_{\mathbb{Z}}(_, \mathbb{Q})$, $F_4 = Hom_{\mathbb{Z}}(\mathbb{Z}/5\mathbb{Z}, _)$.
- (2) Suppose that a square matrix A admits a Jordan normal form. Prove that the transpose of A also admits a Jordan normal form and the two normal forms are equivalent.
- (3) Show that any group of order p^n is solvable, i.e. has an abelian tower whose last element is $\{e\}$.
- (4) Let R be a ring. Consider the following short exact sequence of left R -modules:

$$0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0.$$

Let F be a free R -module and let $\alpha : F \rightarrow C$ be any R -module homomorphism. Show that there exist an R -module homomorphism $\beta : F \rightarrow B$ such that $\alpha = f \cdot \beta$. (Remark: this is called projective property of the free module F .)

- (5) Let G be a group of order $1798 = 31 \cdot 29 \cdot 2$ acting on a set of 27 elements. Then G has a fixed point.
- (6) Consider the ring $\mathbb{Z}/81\mathbb{Z}$. Show that the group of units of this ring is cyclic.
- (7) Consider the following homomorphism of \mathbb{Z} -modules:

$$(\mathbb{Q}/\mathbb{Z})(p) \xrightarrow{f} (\mathbb{Q}/\mathbb{Z})(p),$$

where $(\mathbb{Q}/\mathbb{Z})(p)$ is the p -primary subgroup of \mathbb{Q}/\mathbb{Z} , and the map f is multiplication by p . Describe $Ker(f)$, $Im(f)$ and $Coker(f)$.

- (8) Let $f : R \rightarrow S$ be a surjective homomorphism of rings, and assume that R is local and $S \neq 0$. Show that S is local.
- (9) Give an example of a ring which does not have unique factorization.

- (10) TRUE/ FALSE (Either prove, or state the theorem that you are using, or give a counterexample and explain.)
- (a) Every torsion group is finite.
 - (b) A 3-Sylow subgroup of the group of permutations on 3 elements is normal.
 - (c) $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \simeq \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
 - (d) Every subgroup of a free abelian group is free.
 - (e) Let $(\mathbb{Q}/\mathbb{Z})(5)$ be the 5-primary subgroup of \mathbb{Q}/\mathbb{Z} . The group $\mathbb{Z}/5\mathbb{Z}$ is a subgroup of $(\mathbb{Q}/\mathbb{Z})(5)$.