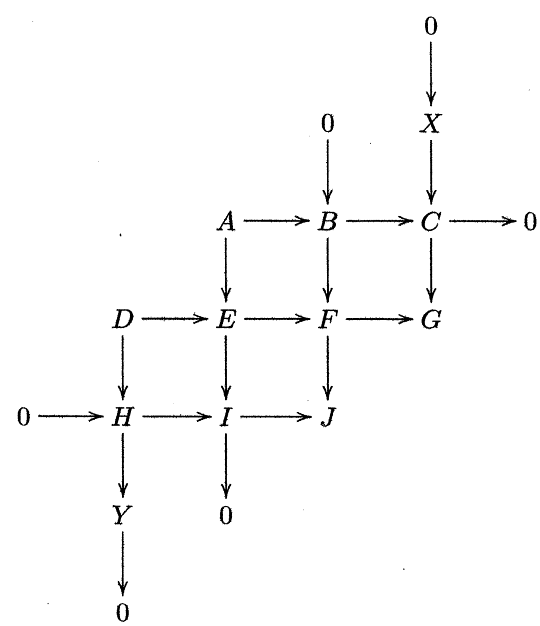
Northeastern University
Mathematics Department

Qualifying Exam, General Algebra
Fall 2007

(1) The Fibonacci series is a sequence beginning with 1, 1 and such that each subsequent term is the sum of the two preceding ones. Prove that the $(n+1)$st term of the Fibonacci series equals the following determinant of order n:

$$A_n = \begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{vmatrix}$$

(2) The following is a commutative diagram



in the category of vector spaces over a field with exact rows and columns.
(a) Use diagram chase to construct a homomorphism $X \to Y$.
(b) Use diagram chase to show that the constructed map is an epimorphism.
(c) Use diagram chase to show that the constructed map is a monomorphism.

(To make your proof shorter and more readable, redraw the diagram and use sequentially labeled arrows in place of your argument. If a further explanation is required in regard to certain arrows, provide a verbal argument referring to the corresponding labels.)

(3) Let $f : V \to V$ be an endomorphism of a 4-dimensional vector space such that $f^2 = f$. Describe all possible Jordan normal forms of f.

(4) Let $u_1, \ldots, u_{2007}, v_1, \ldots, v_{1492}$ be a basis of a vector space V over a field k. Let $f : V \to V$ be the endomorphism such that:

$$\begin{aligned} f(u_i) &= u_{i+1} \quad \text{for} \quad i = 1, \ldots, 2006 \\ f(v_i) &= v_{i+1} \quad \text{for} \quad i = 1, \ldots, 1491 \\ f(u_{2007}) &= 0 \\ f(v_{1492}) &= v_{2007} \end{aligned}$$

a) What are the eigenvalues of f?
b) Determine the Jordan normal form of f (in particular, your result will show that f does have a Jordan normal form).

(5) Prove that any linear function f on the space $M_n(k)$ of all square matrices of order n over a field k is of the form

$$f(X) = \operatorname{tr} AX,$$

where tr denotes the trace and A is a square matrix uniquely determined by f.

(6) Let G be the group of permutations on 7 elements. Prove that the 7-Sylow subgroup is not normal.

(7) Prove that any group of order 125 has more then 10 conjugacy classes.

(8) Consider the abelian group $\mathbb{Q}/\mathbb{Z}$. Does it have a subgroup of order 27? Explain your answer.

(9) Find $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z})$. Prove your statement.

(10) Describe the kernel, image and cokernel of the homomorphism $f : \mathbb{Z}/100\mathbb{Z} \to \mathbb{Z}/20\mathbb{Z}$ given by $f(x) = 2x$.