NORTHEASTERN UNIVERSITY
DEPARTMENT OF MATHEMATICS

QUALIFYING EXAM IN ANALYSIS, PART I
APRIL 2005

1. (a) For x in $(0,\infty)$, let $h(x) = \sin(1/x)$. Is $h(x)$ uniformly continuous on $(0,\infty)$? Is it uniformly continuous on (a,∞) for $a > 0$?

(b) Prove that, if f is uniformly continuous on a bounded set $S \subset \mathbb{R}^n$, then f is bounded on S.

2. Let $BC^1(\mathbb{R})$ denote the set of bounded continuous functions on $\mathbb{R}$ whose derivative is bounded and continuous. For $f, g \in BC^1(\mathbb{R})$ set

$$\rho(f,g) \;=\; \sup_{x\in\mathbb{R}} |f(x) - g(x)| \;+\; \sup_{x\in\mathbb{R}} |f'(x) - g'(x)|.$$

Does ρ define a metric on $BC^1(\mathbb{R})$?

3. Consider the function $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}$.

(a) For which values of x does the series converge absolutely?

(b) On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly?

(c) Is f continuous wherever the series is convergent?

4. Show that a solution of the initial value problem

$$\begin{aligned} y'(x) &= \frac{1}{3+\sin x + y^2}, \\ y(0) &= 1 \end{aligned}$$

exists for all $x \geq 0$.

Hint: Use the contraction principal to show that if a solution $y(x)$ exists on the interval $[0,T]$ then it also exists on the interval $[0, T+1]$.

5. Show that the function

$$f(x) \;=\; \begin{cases} \sin(1/x) & \text{if} \quad x \neq 0; \\ 0 & \text{if} \quad x = 0 \end{cases}$$

is Riemann integrable on any interval $[a,b]$.

6. Consider the one-form

$$\eta = \frac{xdy - ydx}{x^2 + y^2}$$

on $\mathbb{R}^2 - \{0\}$.

(a) Show that $d\eta = 0$.

(b) Let

$$\gamma(t) = (\cos 3t, \sin 3t) \in \mathbb{R}^2, \qquad t \in [-\pi, \pi].$$

Find $\int_\gamma \eta$.