

Northeastern University, Mathematics Department.

Analysis 2: Qualifying Exam

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Be sure to include your reasoning – no credit for unexplained answers.

1.) Let $\langle E_i \rangle$ be a sequence of disjoint measurable sets and A any set. Show that

$$m^*(A \cap \bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$$

2.) A function f is said to satisfy a Lipschitz condition on an interval if there is a constant M such that $|f(x) - f(y)| \leq M|x - y|$ for all x and y in the interval.

a) Show that a function satisfying a Lipschitz condition is absolutely continuous.

b) Show that an absolutely continuous function f satisfies a Lipschitz condition if and only if $|f'|$ is bounded.

c) Show that f satisfies a Lipschitz condition if one of its derivatives is bounded (for example $D^+ f(x) = \limsup_{h \rightarrow 0^+} \{f(x+h) - f(x)\}/h$)

3.) Prove that there exists a Riemannian structure on every differentiable manifold. [Hint: you need to use a partition of unity argument].

4.) Derive the Fourier series

$$\frac{t}{2} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(nt)}{n}, \quad -\pi < t < \pi$$

5.) For $1 \leq p < \infty$, we denote by l^p the space of all sequences $\langle \xi_\nu \rangle_{\nu=1}^{\infty}$ such that $\sum_{\nu=1}^{\infty} |\xi_\nu|^p < \infty$. Prove the Minkowski inequality for sequences:

$$\| \langle \xi_\nu + \eta_\nu \rangle \|_p \leq \| \langle \xi_\nu \rangle \|_p + \| \langle \eta_\nu \rangle \|_p$$