Northeastern University Department of Mathematics

Qualifying Exam: Analysis 1

April 2006

Note: Use blue books. Be sure to include the reasoning behind your answers. You may use well-known results without proving them, but state such results clearly.

1. Consider the following series
$$\sum_{n=0}^{\infty} \frac{x^n}{1+x^{2n}}$$
   (a) For what real values of x does this series converge?
   (b) For which intervals I is the series uniformly convergent?

2. Let I be an interval and fix $x \in I$. Suppose that $f_n : I \to \mathbf{R}$ is continuous at x for each $n \in \mathbf{N}$, and the sequence $\{f_n\}$ converges uniformly on I to f. If $x_n \in I$ satisfies $x_n \to x$ as $n \to \infty$, show that $f_n(x_n) \to f(x)$ as $n \to \infty$.

3. Suppose that $f(x)$ is defined and differentiable for every $x > 0$ and $f'(x) \to 0$ as $x \to \infty$. Let $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to \infty$.

4. Suppose that f is continuous on $[0, \infty)$ and $f(x) \to L$ as $x \to \infty$. Show that, for any $a > 0$, $\lim_{n\to\infty} \int_0^a f(nx)\,dx = aL$.

5. Let X and Y be metric spaces and $f : X \to Y$ be continuous. (a) If X is compact, show that $f(X)$ is compact. (b) If X is connected, show that $f(X)$ is connected.

6. Let $\vec{F} : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $\vec{F}(x, y) = (e^x \cos y, e^x \sin y)$.
   (a) Is $\vec{F}$ one-to-one on $\mathbf{R}^2$?
   (b) Is $\vec{F}$ onto $\mathbf{R}^2$?
   (c) Is $\vec{F}$ locally invertible? (Explain what this means.)
   (d) What are the images of lines that are parallel to the coordinate axes?