

NORTHEASTERN UNIVERSITY
DEPARTMENT OF MATHEMATICS

ANALYSIS 1 QUALIFYING EXAM

JANUARY 2009

1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function. A real number $x \in \mathbb{R}$ is called a *fixed point* of f if $f(x) = x$. Show that if $f'(t) \neq 1$ for every real t , then f has at most one fixed point.

2. Suppose that the sequence of functions $\{f_k\}$ converges uniformly to f on the closed interval $[a, b]$ and that each f_k is Riemann-integrable on $[a, b]$. Show that f is Riemann-integrable on $[a, b]$ and that

$$\lim_{n \rightarrow \infty} \int_a^b f_n(t) dt = \int_a^b f(t) dt.$$

Hint: Use the *Riemann criterion of integrability*: A function f is Riemann-integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition π_ε of $[a, b]$ such that for every refinement π of π_ε we have $U(f, \pi) - L(f, \pi) < \varepsilon$, where $U(f, \pi)$ and $L(f, \pi)$ denote the upper and the lower Riemann sums respectively.

3. A metric space is called *separable* if it contains a countable dense subset. Show that if X is a metric space in which every infinite subset has a limit point, then X is separable.

4. Prove that

$$\frac{\pi^2}{12} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^2}.$$

Hint: Consider the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)), \quad x \in [-\pi, \pi]$$

of $f(x) = x^2$. Show that it converges pointwise to x^2 for $x \in (-\pi, \pi)$ and substitute $x = \pi/2$ to the equality

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)), \quad x \in (-\pi, \pi).$$

5. Let ω be a 2-form on $\mathbb{R}^3 \setminus \{0\}$ defined by

$$\omega = \frac{x dy \wedge dz + y dz \wedge dx + z dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

- (a) Show that ω is closed.
- (b) Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ be the unit sphere in \mathbb{R}^3 . Compute $\int_S \omega$.
- (c) Prove that ω is not exact.