

Qualifying Exam: Analysis 1
January 2006

Note: Use blue books. Be sure to include the reasoning behind your answers. You may use well-known results without proving them, but state such results clearly.

1. Does the following series converge uniformly for $|x| \leq 1$?

$$\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$$

2. Let $f : (0, 1) \rightarrow \mathbf{R}$ be uniformly continuous. Show that $\lim_{x \rightarrow 0^+} f(x)$ exists.

3. Suppose $f : (a, b) \rightarrow \mathbf{R}$ is differentiable with $f' > 0$.

a. Prove that f is strictly increasing on (a, b) .

b. Show that $g = f^{-1}$ is differentiable and $g'(f(x)) = (f'(x))^{-1}$ for $x \in (a, b)$.

4. Suppose f and g are continuous on $[a, b]$ with $g \geq 0$. Show that there exists $c \in (a, b)$ such that

$$\int_a^b f(x)g(x) dx = f(c) \int_a^b g(x) dx.$$

5. Let X be a metric space in which every infinite subset has a limit point. Show that X is separable.

6. Consider the system of equations

$$x + y + z^2 = 0$$

$$x - y - z = 0.$$

a. Find the values $(x, y, z) = (a, b, c)$ for which these equations (locally) determine x and y as functions of z .

b. Find values of $(x, y, z) = (a, b, c)$ for which these equations (locally) determine x and z as functions of y .