NORTHEASTERN UNIVERSITY
Department of Mathematics

Qualifying Exam in Analysis, Part I
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For full credit you should provide proofs for all statements which you make, except the ones which can be found in the standard textbooks, which are quoted in the syllabus of the exam.

1. Denote by $C_{pl}([0,1])$ the space of all continuous piecewise linear functions on $[0,1]$ (that is, continuous functions $f : [0,1] \to \mathbf{R}$, satisfying the following condition: there exists a partition $\mathcal{P} = \mathcal{P}(f)$ of $[0,1]$, such that f is linear on every interval of this partition.)

(a) Introduce the following distance on $C_{pl}([0,1])$:

$$\rho(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|, \quad f,g \in C_{pl}([0,1]).$$

Prove that with this distance function, $C_{pl}([0,1])$ becomes a metric space.

(b) Show that this metric space is incomplete and find its completion.

(c) For a a fixed function $f \in C_{pl}([0,1])$, describe the set of points $x \in [0,1]$ where there exists derivative $f'(x)$.

(d) Prove that the integration by parts formula

$$\int_0^1 f'(x)g(x)dx = f(x)g(x)|_{x=0}^{x=1} - \int_0^1 f(x)g'(x)dx$$

holds for all $f,g \in C_{pl}([0,1])$.

(e) Does it hold if we remove the continuity requirement for f,g (but require only linearity of f,g on any open interval (a_i, a_{i+1}) of a given partition $0 = a_0 < a_1 < \ldots < a_N = 1$)?

2. Consider a differential equation

$$y'' + q(x)y = 0,$$

where $x \in \mathbf{R}$ is the independent variable, $y = y(x)$ is the unknown function, $q = q(x)$ is a given real-valued function on $\mathbf{R}$, $q \in C(\mathbf{R})$.

(a) Prove that the real-valued solutions y form a linear space with respect to the usual addition of functions and multiplication of functions by a number.

(b) What is the dimension of this space of solutions?

(c) Suppose that $y_1(x)$ satisfies the equation with initial condition $y_1(0) = 0$. Prove or give a counterexample: $y_1(x) = 0$ for all x.

(d) Prove that the set of all solutions satisfying the initial condition $y(0) = 0$ forms a linear space. What are the possible dimensions of this space? Assuming that there is a nonzero solution y_1 in this set, how can you describe this space in terms of y_1?

(e) Fix two points $a, b \in \mathbf{R}$, $a \neq b$, and consider the set of solutions, such that $y(a) = y(b) = 0$. Prove that this set is a linear space. Which values can the dimension of such a space have?

3. Find all functions $f : \mathbf{R}^2 \to \mathbf{R}$, $f = f(x, y) \in C^2(\mathbf{R}^2)$, such that

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0,$$

at all points in $\mathbf{R}^2$.

4. Consider the map $f : \mathbf{R}^3 \to \mathbf{R}^3$ given by the formula

$$f(x, y, z) = (x - y^2 z, 2y + \sin z, 3z).$$

Let B be the unit ball in $\mathbf{R}^3$ centered at the origin.

(a) Prove that the map f is a diffeomorphism $\mathbf{R}^3 \to \mathbf{R}^3$.

(b) Find the volume of $f(B)$.