

Mathematics Department, Northeastern University

Analysis 1 Qualifying Exam, September 2008.

Note: use blue books. Answer all questions. Be sure to include the reasoning behind your answers. You may use well-known results without proving them, but state such results clearly.

1.) a). Let (X, d) be a metric space. Provide the definition of the statement: X is compact.

b). Let A_1, A_2, \dots be a decreasing sequence of nonempty closed subsets of a compact metric space, so that $A_1 \supset A_2 \supset \dots$.

Prove or give a counterexample: the intersection $A_1 \cap A_2 \cap \dots$ is non-empty.

2.) a). Let $f : X \rightarrow Y$ where (X, d) and (Y, d') are metric spaces. Provide the definition of the statement: f is uniformly continuous.

b). Prove or disprove: the function $f(x) = \sqrt{|x|}$ is uniformly continuous on \mathbf{R} .

[Hint: consider separately the two sets $[-1, 1]$ and $\mathbf{R} \setminus [-1, 1]$].

3.) a). Let $f : U \rightarrow \mathbf{R}$ where $U \subset \mathbf{R}$ is open. Provide the definition of the statement: f is differentiable on U .

Let $f : [a, b] \rightarrow \mathbf{R}$ be continuous, and differentiable on (a, b) .

b). State the mean value theorem for f .

c). Suppose in addition that $f'(x)$ is bounded in (a, b) . Prove or give a counterexample: f is uniformly continuous.

4.) a). Define the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n (x - a)^n$.

b). Find the radius of convergence of the power series $\sum n^{-n/2} x^n$.

5.) Let $f : [0, \infty) \rightarrow \mathbf{R}$ be continuous, and satisfy $\lim_{x \rightarrow +\infty} f(x) = c$. Prove or give a counterexample:

$$\lim_{x \rightarrow +\infty} \frac{1}{x} \int_0^x f(t) dt = c$$

6.) a). Let $f : X \rightarrow X$ where (X, d) is a metric space. Provide the definition of the statement: f is a contraction, and p is a fixed point for f .

b). Prove that the equation $1 - \sin x = x$ has a unique solution. Compute this solution to three decimal places.

7.) Let $f(x, y) = x + y$, and $g(x, y) = xy + \cos x$.

a). Identify and draw the set of points in \mathbf{R}^2 at which it is not possible to use the Inverse Function Theorem to find two locally defined functions u and v satisfying

$$u(f(x, y), g(x, y)) = x, \quad v(f(x, y), g(x, y)) = y.$$

b). Find a real number a such that the Inverse Function Theorem can be applied at the point $P = (a, 0)$ in \mathbf{R}^2 . Calculate the following four partial derivatives of the local inverse functions u and v at the point $(f(P), g(P))$:

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y}$$

(your answers should be real numbers).