Northeastern University Department of Mathematics

# Qualifying Exam: Analysis 1

## September 2006

*Note: Use blue books. Be sure to include the reasoning behind your answers. You may use well-known results without proving them, but state such results clearly.*

1. Consider the following series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$$

   (a) For what real values of $x$ does this series converge?
   (b) For what real values of $x$ is the series absolutely convergent?
   (c) Show that the series is uniformly convergent on bounded sets of $\mathbf{R}$.

2. (a) Give an example of a bounded continuous function on a bounded interval that is not uniformly continuous.
   (b) Give an example of a differentiable function $f$ such that $f'$ is not continuous.

3. If $f$ is is differentiable and $f'(0) > 0$, show that there is a $\delta > 0$ such that $f(-x) < f(0) < f(x)$ for all $0 < x < \delta$.

4. (a) Assume that $f$ is continuous on $[0,1]$. Show that

$$\int_0^{\pi/2} f(\cos x)\,dx = \int_0^{\pi/2} f(\sin x)\,dx.$$

   (b) Assume that $f$ is $C^1$ on $[0,1]$ and $f(1) = 0$. Show that

$$\int_0^1 x f(x) f'(x)\,dx = -\frac{1}{2}\int_0^1 |f(x)|^2\,dx.$$

5. (a) Define a metric space.
   (b) If $(X, d)$ is a metric space and $\{x_n\}$ is a sequence satisfying $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} x_n$, then show $x = y$.
   (c) Give an example of a topological space $Y$ for which there is a sequence $\{x_n\}$ satisfying $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} x_n$ but $x \neq y$.

6. Let $f : \mathbf{R}^2 \to \mathbf{R}^3$ by $f(x,y) = (y^2, x^2, x+y)$ and $g : \mathbf{R}^2 \to \mathbf{R}^2$ by $g(x,y) = (y\cos x, x\sin y)$.
   (a) Show that $g$ is a bijection on a neighborhood $U$ of $(0, \pi/2)$.
   (b) Let $h(x,y) = f \circ g^{-1}(x,y)$ on $g(U)$. Compute the matrix $dh(\pi/2, 0)$.