Qualifying Exam: Analysis 1

September, 2005

Note: Be sure to include the reasoning behind your answers. You may use well-known results without proving them, but state such results clearly.

1. (a) For a sequence $(x_n)_{n=1}^\infty$, state the definition of convergence.
   (b) State the Cauchy condition for $(x_n)_{n=1}^\infty$.
   (c) Show that a bounded, monotone (increasing or decreasing) sequence is convergent.
   (d) Let $x_1 = 1$ and $x_{n+1} = \sqrt{1+x_n}$ for $n \geq 1$. Show that the sequence $(x_n)_{n=1}^\infty$ converges as $n \to \infty$ and find its limit.

2. (a) Consider
$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n+1}{2n(n+1)}$$
   Show the series converges and find its limit. Does the series converge absolutely?
   (b) Consider the series
$$\sum_{n=2}^{\infty}\frac{x^n}{n\log n}$$
   For which values of x does the series converge and for which does it diverge?

3. Consider a sequence of functions $(f_n)_{n=1}^\infty$ on an interval (a,b).
   (a) Define uniform convergence for f_n on (a,b).
   (b) Let $(a,b) = (0,1)$ and $f_n(x) = \frac{\log(nx)}{n}$. Show $f_n(x) \to 0$ for each $x \in (0,1)$.
   (c) Does $f_n \to 0$ uniformly on $(0,1)$? Justify your answer.

4. (a) Use Riemann sums to calculate
$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2}$$
   (b) Let $f_n(x) = ne^{x^2}/(1+n^2)$. What is $\lim_{n\to\infty}\int_{-1}^{1} f_n dx$?
   (c) Use substitution $u = x^2$ and integration by parts to show that $\lim_{R\to\infty}\int_1^R \cos(x^2)\,dx$ is finite.

5. Let $\mathbf{R}_+^2 = \{(x,y) \in \mathbf{R}^2 : y > 0\}$ and $F : \mathbf{R}_+^2 \to \mathbf{R}^2$ be defined by $F(x,y) = \begin{pmatrix} x/y \\ x^2 - y^2 \end{pmatrix}$.
   (a) Determine the regions in $\mathbf{R}_+^2$ for which F is invertible.
   (b) Note that $F(0,1) = (0,-1)$. Find F^{-1} in a neighborhood of $(0,-1)$.
   (c) Is $F : \mathbf{R}_+^2 \to \mathbf{R}^2$ surjective (onto)? If not, describe the image of F.