Northeastern University
(Department of Mathematics)

Analysis 2 Qualifying Exam (Spring 2006)

1. Let f be an absolutely continuous function on $(0,1)$. Prove that for any $\varphi \in C_0^\infty(\mathbb{R})$,

$$\int_0^1 f'(x)\varphi(x)\,dx = -\int_0^1 f(x)\varphi'(x)\,dx\,.$$

2. Let $0 < \omega < 1$ be an irrational number. Define the equivalence relation on $I := [0,1)$

$$x \sim y \iff \exists k \in \mathbb{Z} \text{ such that } x \equiv y + k\omega (\bmod 1).$$

Denote by $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ the equivalence classes of I with respect to $\sim$ and for any $\alpha \in \mathcal{A}$ choose a unique representative $x_\alpha \in I_\alpha$. Prove that the set $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ is not Lebesgue measurable.

3. (a) Prove that the restriction of the 2-form

$$\alpha = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$$

to the unit sphere $\{x^2+y^2+z^2=1\}$ is not exact.
(b) Prove that the 2-form

$$\beta = \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2+y^2+z^2)^{3/2}}$$

is closed on $\mathbb{R}^3 \setminus \{0\}$ but not exact.

Let μ be the Lebesgue measure on $\mathbb{R}$.

4. Let $K(x,y)$ be a square integrable function on $[0,1]\times[0,1]$. Prove that for any $f \in L^2([0,1])$ there exists a subset $A \subseteq [0,1]$, $\mu(A) = 1$, such that for any $x \in A$ the function $y \mapsto K(x,y)\,f(y)$ is Lebesgue integrable and

$$(Kf)(x) := \int_0^1 K(x,y)\,f(y)\,dy$$

is square integrable on $[0,1]$.

5. Let f be a 1-periodic function on $\mathbb{R}$ that is square integrable on the interval $[0,1]$. Assume that the Fourier coefficients $\hat{f}_n$ of f satisfy the inequality

$$\sum_{k\in\mathbb{Z}} |\hat{f}_k|^2 |k|^{2s} < \infty$$

with $s > 1/2$. Prove that f is continuous.

6. (Extra credit) Find the Fourier series of $f(x) = \cos^3 x$ on $[0, 2\pi]$.