Mathematics Department, Northeastern University

Analysis 1 Qualifying Exam, May 2008.

Note: use blue books. Answer all questions. Be sure to include the reasoning behind your answers. You may use well-known results without proving them, but state such results clearly.

1.) *a).* Provide the definition of the statement: d is a metric on the set X.

b). Let (X, d) be a metric space. For $x, y \in X$ define

$$r(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

Prove or give a counterexample: r is a metric on X.

2.) Let (X, d) be a metric space, let K be a subset of X, and let $x \in X$.

a). Provide the definition of the statement: K is a compact subset of X.

b). Provide the mathematical definition of $d(x, K)$, the distance from x to K.

c). Assuming that K is compact, show that there is $y \in K$ such that $d(x, K) = d(x, y)$.

3.) *a).* Let $f : U \to \mathbf{R}$ where $U \subset \mathbf{R}$ is open. Provide the definition of the statement: f is differentiable on U.

b). Define

$$f(x) = \begin{cases} x^a \sin(x^{-c}) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Find necessary and sufficient conditions on the positive numbers a, c which guarantee that $f'(x)$ is continuous.

4.) Let

$$f_n(x) = \sum_{k=0}^{n} xe^{-kx}.$$

a). Prove that $\{f_n\}$ converges uniformly on the interval $0 \leq x \leq 1$, as $n \to \infty$. What is the limit?

b). Justify the following formula:

$$\int_0^1 \frac{x}{1-e^{-x}}\,dx = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(n+1)e^{-n}}{n^2}.$$

5.) Let $f : [0,\infty) \to \mathbf{R}$ be continuous, and satisfy $\lim_{x\to+\infty} f(x) = 0$. Prove or give a counterexample:

$$\lim_{x\to+\infty} \frac{1}{x}\int_0^x f(t)\,dt = 0$$

6. Prove that the equation $1 - \sin x = x$ has a unique solution. Compute this solution to three decimal places.

7.) Let $f(x,y) = x + y^2 - 1$, and $g(x,y) = 2xy$.

a). Identify and draw the set of points in $\mathbf{R}^2$ at which it is not possible to use the Inverse Function Theorem to find two locally defined functions u and v satisfying

$$u(f(x,y), g(x,y)) = x, \quad v(f(x,y), g(x,y)) = y.$$

b). Find a real number a such that the Inverse Function Theorem can be applied at the point $P = (a,0)$ in $\mathbf{R}^2$. Calculate the following four partial derivatives of the local inverse functions u and v at the point $(f(P), g(P))$:

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y}$$

(your answers should be real numbers).

5. Let $B = S^1 \vee S^1$ be the wedge of two circles, and choose the wedge point b_0 as basepoint.

(a) Define a two-fold covering map $p\colon E \to B$, with E connected. Pick $e_0 \in p^{-1}(b_0)$.

(i) Find a subgroup H of $\pi_1(B, b_0)$ corresponding to p. Show that H is a normal subgroup. Describe the group of deck transformations of p.

(ii) Determine the induced homomorphism $p_\sharp\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$.

(iii) Determine the induced homomorphism $p_*\colon H_1(E; \mathbb{Z}) \to H_1(B; \mathbb{Z})$.

(b) Let Y be the real line to which circles are attached at all integer points. Show that there exists a regular covering map $p\colon Y \to B$. Find a subgroup of $\pi_1(B, b_0)$ corresponding to that cover, and describe the group of deck transformations.