Department of Mathematics
Combinatorics Qualifying Exam
April 2007

1. Let G be a connected graph.

   (a) Prove that any two longest paths have a common vertex.

   (b) Let G be a k-connected graph, and suppose that G' is obtained from G by adding a new vertex y with k neighbors in G. Prove that G' is k-connected.

2. Let G be a connected bipartite graph.

   (a) Prove that the bipartition of G is unique.

   (b) Suppose that the bipartition of G is (X, Y) and $|X| \geq |Y|$.
   Prove: Every independent subset of $V(G)$ has cardinality $\leq |X|$ if and only if G has a matching M which saturates the vertices of Y.

3. Consider a rectangular box P with length 1, width 2 and height 3. Color the vertices of P with k colors, but do not distinguish two colorings which are equivalent under the symmetry group G of P.

   (a) Find the cycle indicator polynomial of G.

   (b) What is the total number of models (equivalence classes of colorings) obtained?

4. Let $A_1, \ldots, A_n$ be a family of nonempty sets. Show that there exists a family $B_1, \ldots, B_n$ of mutually disjoint 2-element sets with $B_i \subseteq A_i$ for $i = 1, \ldots, n$, if and only if

   $$|A_{j_1} \cup \ldots \cup A_{j_k}| \geq 2k \qquad (k \leq n;\ 1 \leq j_1 < \ldots < j_k \leq n).$$

   (Hint: Apply the Marriage Theorem to a suitable family of sets. Remember it is not forbidden to have a set appear more than once as member of a family. Consider what this would mean for a system of representatives.)

5. The vectors $v = (1, 1)$ and $w = (-1, 1)$ span a sublattice Λ of $\mathbb{Z}^2$. For $t > 0$ define $\Lambda_t := t\Lambda$. Let C be the square centered at the origin and with vertices at $(1, 1)$, $(-1, 1)$, $(-1, -1)$ and $(1, -1)$.

   (a) For which values of t is Λ_t admissible for C?

   (b) For which values of t does Λ_t pack C (that is, for which t is $\Lambda_t + C$ a packing of $\mathbb{E}^2$)? What is the density of these packings of squares?

6. Recall that the dual lattice Λ^* of a lattice Λ in $\mathbb{E}^d$ is given by

$$\Lambda^* := \{x \in \mathbb{E}^d \mid x \cdot y \in \mathbb{Z} \text{ for all } y \in \Lambda\}.$$

   Find Λ^* when Λ is the hexagonal lattice in the plane spanned by two unit vectors inclined at $60°$.