

Northeastern University  
Mathematics Department  
Qualifying Exam, General Algebra, Fall 2008

1. Let  $A$  be a  $n \times n$  matrix with entries in a field  $K$  such that  $A^2 = I_n$ . Show that  $\text{rank}(I_n + A) + \text{rank}(I_n - A) = n$ .

2. Let  $A$  be a nilpotent matrix (with entries in a field  $K$ ), i.e., a square matrix such that  $A^r = 0$ , for some positive integer  $r$ . Find the characteristic polynomial of  $A$ . Give reasons for your answer.

3. Consider the following quadratic form over  $\mathbb{R}$ :

$$Q(x_1, x_2, x_3, x_4) = 2x_1x_2 + 2x_1x_3 + x_2^2 + 2x_2x_4 + x_3^2 + 2x_3x_4 + x_4^2.$$

Find the rank and signature of  $Q$ .

4. Describe all conjugacy classes of  $6 \times 6$  matrices  $A$  with minimal polynomial  $M_A = x^2(x + 8)$ , and characteristic polynomial  $P_A(x) = x^4(x + 8)^2$ .

5. Let  $f : K^3 \rightarrow K^2$  be a linear map with the matrix

$$A = \begin{pmatrix} 4 & -1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$$

in the co-ordinate bases. Find the kernel and image of the linear map

$$\Lambda^2 f : \Lambda^2 K^3 \rightarrow \Lambda^2 K^2$$

6. Prove that a group of order 125 has nontrivial center.

7. Describe up to isomorphism all abelian groups of order 300.

8. Prove that any group of order  $10 \cdot 11^5$  is not simple.

9. Let  $G$  be the group of permutations on 7 elements. Show that the 7-Sylow subgroup is not normal.

10. Describe  $\text{Hom}(\mathbb{Z}/5\mathbb{Z}, \mathbb{Z})$ , the set of all group homomorphisms from the group  $\mathbb{Z}/5\mathbb{Z}$  to the group  $\mathbb{Z}$ .