QUALIFYING EXAM, GENERAL ALGEBRA, April 2005

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1. Describe all possible Jordan canonical forms of 5x5 matrices A satisfying $(A-4I)^3(A+2I)=0$.

2. Let $f:\mathbb{R}^2 \to \mathbb{R}^3$ be a linear operator given by the matrix: $\begin{bmatrix} 4 & -1 \\ 5 & 0 \\ 2 & 5 \end{bmatrix}$.

   Find the kernel and the image of the operator:

   $$\wedge^2 f : \wedge^2(\mathbb{R}^2) \to \wedge^2(\mathbb{R}^3).$$

3. Consider the following quadratic form over $\mathbb{R}$:

   $Q(\underline{x}) = 2x_1x_2 + 2x_1x_3 + x_2^2 + 2x_2x_4 + x_3^2 + 2x_3x_4 + x_4^2$.

   Find the rank and signature of Q.

4. Let V be a vector space and V^* its dual. Show that the map:

   $$f : V^* \otimes \wedge^k V \to \wedge^{(k-1)} V,$$

   given by:

   $$f(\phi \otimes v_1 \wedge \ldots \wedge v_k) = \Sigma(-1)^{(j+1)}\phi(v_j)v_1 \wedge \ldots \wedge v_{(j-1)}v_{(j+1)} \wedge \ldots \wedge v_k,$$

   is well defined. (The sum is taken over all $j = 1, \ldots, k$.)

5. a) Prove that any homomorphism between two simple groups is either zero map or is one-to-one map.

   b) Give an example of a homomorphism between two simple groups, which is NOT an isomorphism.

6. Are there any simple groups of order 505? Justify your answer.

7. Describe, up to isomorphism, all abelian groups of order 1000, which do NOT have any elements of order 40.

8. Let p be a prime different from 2, 3 and 5. Let G be a p-group. Consider S_5, the group of permutations of 5 elements. Prove that any action of G on S_5 must have a fixed point.

9. TRUE - FALSE Please, justify your answer - either indicate the proof, or give a counterexample.

   a. Every infinite group has an element of infinite order.

   b. A homomorphism between two groups which is one-to-one must be an isomorphism.

   c. A homomorphism between two vector spaces which is one-to-one must be an isomorphism.

   d. Every nxn matrix over $\mathbb{C}$ is diagonalizable.

   e. If the minimal polynomial of a 5x5 matrix is $T - 8$, the matrix has an eigenbasis.

10. Anything that you studied, or you really like, but we didn't ask? Please, state the problem AND solve it.