Northeastern University
Mathematics Department

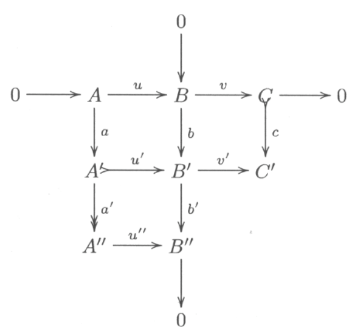
Qualifying Exam, General Algebra
February 2007

(1) The integers $1798 = 31 \cdot 58, 2139 = 31 \cdot 69, 3255 = 31 \cdot 105, 4867 = 31 \cdot 157$ are all divisible by 31. Without any calculations, prove that the determinant

$$\begin{vmatrix} 1 & 7 & 9 & 8 \\ 2 & 1 & 3 & 9 \\ 3 & 2 & 5 & 5 \\ 4 & 8 & 6 & 7 \end{vmatrix}$$

is also divisible by 31.

(2) The commutative diagram



in the category of vector spaces has the following properties: 1) the top row and the middle column are exact, 2) $v' \circ u' = 0$ and $a' \circ a = 0$, 3) c and u' are monomorphisms, 4) a' is an epimorphism. Use diagram chase to show that u'' is a monomorphism. (To make your proof shorter and more readable, redraw the diagram and use sequentially labeled arrows in place of your argument. If a further explanation is required in regard to certain arrows, provide a verbal argument referring to the corresponding labels.)

(3) Let $f : V \to V$ be an endomorphism of an n-dimensional vector space. Prove that $\text{rank} f + \text{rank}(1 - f) = n$

(4) Suppose that a square matrix A admits a Jordan normal form. Prove that the transpose of A also admits a Jordan normal form and the two normal forms are equivalent.

(5) Let V be a finite-dimensional vector space over a field k. In the lectures we constructed a canonical isomorphism $N : V^* \otimes V \to \mathrm{Hom}(V, V)$. Show that the composition

$$(V, V) \xrightarrow{N^{-1}} V^* \otimes V \xrightarrow{C} k,$$

where $C(f \otimes v) := f(v)$ for any $f \in V^*$ and $v \in V$, coincides with the trace.

(6) Show that any group of order p^2 is solvable, i.e. has an abelian tower whose last element is $\{e\}$.

(7) Describe all isomorphism classes of abelian groups of order 600.

(8) Describe $\mathrm{Hom}_{Ab}(\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$. Prove your statement.

(9) Consider the following short exact sequence of abelian groups:

$$0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0.$$

Let F be a free abelian group and let $\alpha : F \to C$ be any group homomorphism. Show that there exist a group homomorphism $\beta : F \to B$ such that $\alpha = f\beta$. (Remark: this is called *projective property* of the group F.

(10) True/ False
(Either prove, state a theorem, or give a counterexample and explain.)
- (a) Every torsion group is finite.
- (b) Every group of prime order is cyclic.
- (c) A 3-Sylow subgroup of the group of permutations on 3 elements is normal.
- (d) $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \simeq \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
- (e) Every subgroup of a free abelian group is free.
- (f) If a group of 17 elements acts on a set of 16 elements, the action must be trivial, i.e. $gx = x$ for all g in the group and all x in the set.
- (g) Let H be a subgroup of a group G of index $[G : H] = 7$. Suppose there exist an element $g \in G \backslash H$, $g \neq e$, such that $gHg^{-1} = H$. Then H is a normal subgroup of G.
- (h) Let $(\mathbb{Q}/\mathbb{Z})(5)$ be the 5-primary subgroup of $\mathbb{Q}/\mathbb{Z}$. The group $\mathbb{Z}/5\mathbb{Z}$ is a subgroup of $(\mathbb{Q}/\mathbb{Z})(5)$.