NORTHEASTERN UNIVERSITY
DEPARTMENT OF MATHEMATICS

Qualifying Exam in Geometry

Winter 2005

Give complete proofs or justifications for each statement you make. Show all your work.

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1. Using the Frobenius theorem, find necessary and sufficient conditions for the local existence and uniqueness of solution $u = u(x, y) \in C^\infty$ of the system

$$\begin{cases} \frac{\partial u}{\partial x} = f(x, y, u) \\ \frac{\partial u}{\partial y} = g(x, y, u) \end{cases}$$

in a neighborhood of $(0, 0)$ in $\mathbb{R}^2$, with an arbitrary initial condition $u(0, 0) = u_0$, assuming that f, g are given C^∞ functions of x, y, u.

2. Find explicitly a basis of left-invariant vector fields on the Lie group $SL(2, \mathbb{R})$.

3. (a) Calculate the Christoffel symbols, curvature tensor, Ricci tensor, scalar curvature and sectional curvature for the metric

$$ds^2 = ydx^2 + dy^2$$

on $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2, y > 0\}$.

(b) Write the equations for the corresponding geodesics.

4. Calculate volume of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the metric

$$ds^2 = (2 + \cos 2\pi y)^2 dx^2 + dy^2 .$$

5. Calculate the integral

$$\int_{\partial\Omega} x dy \wedge dz,$$

where

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : \; x^2 + \frac{y^2}{4} + \frac{z^2}{9} \leq 1\}.$$

6. Can a Riemannian metric on the torus $\mathbb{T}^2$ have everywhere negative curvature?