70 points is a passing grade. **Geometry Qualifying Exam** April 2007

Throughout this exam, the term "manifold" will mean a pure-dimensional manifold (i.e., the manifold has the same dimension at each point).

1) (10 points each)

a) Define a smooth (C^∞) n-dimensional manifold N in terms of charts and atlases, i.e., without reference to an ambient space.

b) Using the definition of smooth manifold from part (a), define a smooth m-dimensional **submanifold** M of a given smooth n-dimensional manifold N.

2) Suppose that M and N are smooth manifolds, as defined in Problem 1. Let $x \in M$.

a) (3 points) Suppose that $f : M \to N$ is a function. Define what it means for f to be **smooth**.

b) (4 points) What is a "derivation" and how is this notion used to define a tangent vector to M at a point x, i.e., what is the definition of a tangent vector as a derivation?

c) (3 points) Suppose that $f : M \to N$ is a smooth map. Define $d_x f(\vec{v})$ (frequently written as $df_x(\vec{v})$), where $\vec{v}$ is a derivation.

3)

a) (6 points) Give the definition of an orientation on a smooth manifold (possibly with boundary). You may assume that it is known what an orientation on a finite-dimensional real vector space is.

b) (6 points) If M is an oriented manifold with boundary, define the induced orientation on ∂M.

c) (8 points) Let $\mathbb{H}^n$ denote the upper half-space $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. The boundary $\partial\mathbb{H}^n$ is a copy of $\mathbb{R}^{n-1}$, and at each $p \in \partial\mathbb{H}^n$, the tangent space $T_p(\partial\mathbb{H}^n)$ has a natural orientation given by the the standard basis vectors $(\vec{e}_1, \ldots, \vec{e}_{n-1})$. However, $\mathbb{H}^n$ has a natural orientation given by the the standard basis vectors $(\vec{e}_1, \ldots, \vec{e}_n)$, and then $\partial\mathbb{H}^n$ can be given the induced orientation (as in part b)).

Exactly when do these two differently-defined orientations on $\partial\mathbb{H}^n$ agree? Prove your claim.

4)

a) (8 points) Give the definition of a Riemannian metric on a smooth manifold.

b) (12 points) Let M be an oriented Riemannian manifold. Define the volume form on M and the volume of M.

5)

a) (10 points) State the differential forms version of Stokes' Theorem. Be sure to include all hypotheses.

b) (10 points) In multivariable Calculus courses, Green's Theorem is usually stated in a form like:

Let D be a simple region in $\mathbb{R}^2$, with boundary ∂D. Suppose that f and g are smooth functions from an open neighborhood of D into $\mathbb{R}$. Then,

$$\int_{\partial D} f(x,y)dx + g(x,y)dy = \int\int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy,$$

where ∂D is oriented counterclockwise.

In light of your answer to a), explain all parts of this statement, i.e., explain what is meant here by "simple region", what orientation is being used on D, why must ∂D be oriented counterclockwise, what differential form is used in Stokes' Theorem to obtain Green's Theorem, and then show that Stokes' Theorem really does yield Green's Theorem.