Northeastern University
(Department of Mathematics)

Geometry Qualifying Exam (Spring 2006)

1. Let $X_1, X_2, ..., X_n$ be smooth pairwise commuting vector fields defined in a neighborhood of $0 \in \mathbb{R}^n$. Assume that $X_1(0), ..., X_n(0)$ are linearly independent. Prove that there exist coordinates $\{(y^1, ..., y^n)\}$ (in a neighborhood of 0) such that $X_k = \frac{\partial}{\partial y^k}$, $k = 1, 2, ..., n$.

2. Let (M^n, g) be a compact oriented Riemannian manifold without boundary. The divergence $\operatorname{div} X$ of a vector field X is a function on M^n defined by the relation $L_X \sigma_g = (div\, X)\, \sigma_g$ where L_X is the Lie derivative with respect to X and σ_g is the volume form on M^n. Prove that $\int_M (div\, X)\, \sigma_g = 0$. Find an expression for $div\, X$ in coordinates.

3. Consider the 1-form $\alpha = dz - p_1 dq_1 - p_2 dq_2$ in $\mathbb{R}^5 = \{(z, p_1, q_1, p_2, q_2)\}$. Prove that for any $p \in \mathbb{R}^5$ the restriction of $(d\alpha)_p$ to the kernel of $\alpha_p \in T^*\mathbb{R}^5$ is non-degenerate.

4. Let G be a Lie group. Prove that the mapping $G \to G$, $g \mapsto g^{-1}$, maps left-invariant vector fields into right-invariant ones (and vise versa).

5. Let $X_1, ..., X_n$ be a basis of left-invariant vector fields on a Lie group G, $\dim G = n$, and let $\omega^1, ..., \omega^n$ be a basis of 1-forms such that $\omega^k(X_l) = \delta^k_l$ $(1 \le k, l \le n)$. Prove the Maurer-Cartan formula, $d\omega^k = -\sum_{ij} c^k_{ij}\, \omega^i \wedge \omega^j$, where (c^k_{ij}) are the structure constants of the algebra of G, $[X_i, X_j] = \sum_k c^k_{ij} X_k$.

6. Let X and Y be a left and a right invariant vector fields on a Lie group G. Assume that $X(e) = Y(e) = \xi$ where e is the unital element of G. Prove that the curve $\mathbb{R} \to \mathbb{R}$, $t \mapsto \exp(t\xi)$, is tangent to both X and Y.