

Throughout this exam, you should use the **non**-embedded definition of smooth ( $C^\infty$ ), i.e., smooth manifolds are defined by atlases of coordinate patches, and tangent vectors are derivations of germs of smooth functions.

Also, throughout this exam, the term “manifold” will mean a pure-dimensional manifold (i.e., the manifold has the same dimension at each point).

1) (10 points each) Suppose that  $f : M \rightarrow N$  is a smooth map between smooth manifolds,  $x \in M$ , and  $\vec{v} \in T_x M$ , i.e.,  $\vec{v}$  is a tangent vector to  $M$  at  $x$ .

- Define  $d_x f(\vec{v})$  (frequently written as  $df_x(\vec{v})$ ). Remember: You are NOT in the embedded situation;  $\vec{v}$  is a derivation.
- Suppose that we have another smooth map between smooth manifolds  $g : N \rightarrow P$ . Prove the chain rule, i.e., prove that  $d_x(g \circ f) = d_{f(x)}g \circ d_x f$ .

2) (5 points each) Let  $f : M \rightarrow N$  and  $g : N \rightarrow P$  be smooth maps between smooth manifolds. Let  $Q$  be a smooth submanifold of  $P$ .

- Define what it means for the map  $g$  to be transverse to  $Q$ .
- If  $g$  is transverse to  $Q$ , what can one conclude about  $g^{-1}(Q)$ ?
- For all  $y \in N$ , let  $\pi_{g(y)}$  denote the canonical map from  $T_{g(y)}P$  to  $T_{g(y)}P/T_{g(y)}Q$ , i.e., the map that sends a vector to its equivalence class. Then,  $g$  is transverse to  $Q$  if and only if what is true about the composition  $\widehat{d_y g} := \pi_{g(y)} \circ d_y g$ ? Prove your claim.
- Use your answer to c) to prove the following:  
Suppose that  $g$  is transverse to  $Q$ . Then,  $g \circ f$  is transverse to  $Q$  if and only if  $f$  is transverse to  $g^{-1}(Q)$ .

3)

- (8 points) Give the definition of a Riemannian metric on a smooth manifold.
- (12 points) Use a partition of unity argument to prove that every smooth manifold can be endowed with a Riemannian metric.

4)

- (6 points) Give the definition of an orientation on a smooth manifold (possibly with boundary). You may assume that it is known what an orientation on a finite-dimensional real vector space is.
- (6 points) If  $M$  is an oriented manifold with boundary, define the induced orientation on  $\partial M$ .
- (8 points) Let  $\mathbb{H}^n$  denote the upper half-space  $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ . The boundary  $\partial\mathbb{H}^n$  is a copy of  $\mathbb{R}^{n-1}$ , and at each  $p \in \partial\mathbb{H}^n$ , the tangent space  $T_p(\partial\mathbb{H}^n)$  has a natural orientation given by the the standard basis vectors  $(\vec{e}_1, \dots, \vec{e}_{n-1})$ . However,  $\mathbb{H}^n$  has a natural orientation given by the the standard basis vectors  $(\vec{e}_1, \dots, \vec{e}_n)$ , and then  $\partial\mathbb{H}^n$  can be given the induced orientation (as in part b)).

Exactly when do these two differently-defined orientations on  $\partial\mathbb{H}^n$  agree? Prove your claim.

5)

a) (10 points) State the differential forms version of Stokes' Theorem. Be sure to include all hypotheses.

b) (10 points) In multivariable Calculus courses, Green's Theorem is usually stated in a form like:

Let  $D$  be a simple region in  $\mathbb{R}^2$ , with boundary  $\partial D$ . Suppose that  $f$  and  $g$  are smooth functions from an open neighborhood of  $D$  into  $\mathbb{R}$ . Then,

$$\int_{\partial D} f(x, y)dx + g(x, y)dy = \iint_D \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy,$$

where  $\partial D$  is oriented counterclockwise.

In light of your answer to a), explain all parts of this statement, i.e., explain what is meant here by "simple region", what orientation is being used on  $D$ , why must  $\partial D$  be oriented counterclockwise, what differential form is used in Stokes' Theorem to obtain Green's Theorem, and then show that Stokes' Theorem really does yield Green's Theorem.