Northeastern University
(Department of Mathematics)

Geometry Qualifying Exam (Spring 2007)

1. Let X and Y be vector fields on a manifold M^n and let α be a 1-form. Prove that $(d\alpha)(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$ where $[X,Y]$ denotes the commutator of the vector fields and $X(f)$ is the directional derivative of the function f in the direction X.

2. Let $H(p,x)$ be a smooth function of the variables $p = (p_1, ..., p_n)$ and $x = (x^1, ..., x^n)$. In the chart $\{(p,x,t)\} \cong \mathbb{R}^{2n+1}$ consider the 1-form

$$\alpha = \sum_{k=1}^{n} p_k \, dx^k + H(p,x)\, dt\,.$$

Compute the kernel of the form $d\alpha$. (Bonus: Using Stokes theorem prove that the vector field

$$X_H = \left(\frac{\partial H}{\partial x^1}, ..., \frac{\partial H}{\partial x^n}; -\frac{\partial H}{\partial p_1}, ..., -\frac{\partial H}{\partial p_n}\right)$$

preserves the form $dp_1 \wedge dx^1 + \cdots + dp_n \wedge dx^n$.)

3. Let ω be a closed non-degenerate 2-form on a compact manifold M^m without boundary. Prove that

(*a*) the manifold has even dimension, $m = 2n$;

(*b*) the form $\omega \wedge \cdots \wedge \omega$ (n-times) does not vanish on M^{2n};

(*c*) the manifold M^{2n} is orientable;

(*d*) de Rham groups $H^{2k}_{DR}(M^{2n})$ $(k = 1, 2, ..., n)$ are non-trivial.

4. Let X be a left-invariant vector field on a Lie group G. Prove that for any right-invariant vector field Y, $[X,Y] = 0$.

5. Prove that the group $GL(n, \mathbb{C})$ is connected.