NORTHEASTERN UNIVERSITY
DEPARTMENT OF MATHEMATICS

GEOMETRY QUALIFYING EXAM

JANUARY 2010

1. Let M be a smooth compact manifold.
(a) Let X and Y be vector field on M. Define the commutator $[X, Y]$. (You should show that this commutator is a vector field).
(b) Suppose in certain coordinate chart $X = \sum_{i=1}^{n} a^i(x)\frac{\partial}{\partial x^i}$ and $Y = \sum_{i=1}^{n} b^i(x)\frac{\partial}{\partial x^i}$. Compute $[X, Y]$ in this coordinate chart (you should derive this formula from the definition in part (a) and show all your work).

2. Consider the two dimensional sphere

$$S^2 \;=\; \{(x^1, x^2, x^3) :\in \mathbb{R}^3\, (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}.$$

(a) Show that the vector field $X = x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}$ is tangent to S^2.
(b) Let $F : S^2 \to \mathbb{R}^2$ is given by the formula

$$F(x^1, x^2, x^3) \;=\; (x^1x^2, (x^3)^2).$$

Compute $F_*(X)$.
(c) Compute

$$\int_{S^2} x^1 dx^2 \wedge dx^3$$

3. Let $M = \mathbb{R}^2$ and let $E = M \times \mathbb{R}$ be a trivial line bundle over M. Then the space of sections of E is identified with the space of smooth functions $C^\infty(\mathbb{R}^2)$. Consider the connection on E whose covariant derivative $\nabla : \Gamma(E) = C^\infty(\mathbb{R}^2) \to \Omega^1(M, E) = \Omega^1(M)$ is given by the formula

$$\nabla s \;=\; ds \;+\; s(xdy - ydx).$$

(a) Define the notion of the curvature of the connection ∇.
(b) Compute the curvature of this connection.
(c) fix a number $r > 0$. Compute the holonomy of this connection along the curve $\gamma(t) = (r\cos t, r\sin t)$, $0 \le t \le 2\pi$.

4. Let G be a Lie group.

(a) Define the structure of a Lie algebra on the tangent space T_eG to G at the identity element e.

(b) Define the exponential map $\exp : T_eG \to G$. Show that it is smooth and defines a diffeomorphism from some neighborhood U of 0 in T_eG to $\exp(U)$.

5. Let (M, g) be a Riemannian manifold. Define the Riemannian distance $d(x, y)$ between the points $x, y \in M$. Show that this defines a structure of a metric space on M (you need to show that d is symmetric, satisfies the triangle inequality, and that $d(x, y) = 0$ implies $x = y$).