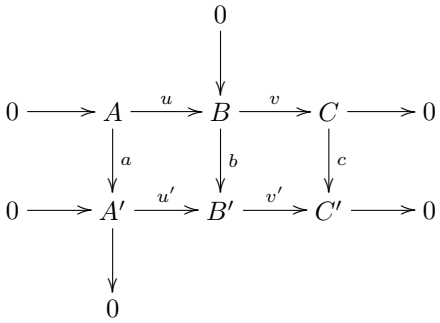
Northeastern University
Mathematics Department

Qualifying Exam, General Algebra
April 2007

(1) Describe all matrices A over $\mathbb{C}$ which have characteristic polynomial $P_A(x) = (x+3)^2(x-5)^4$ and minimal polynomial $M_A(x) = (x+3)^2(x-5)^2$.

(2) Consider the following commutative diagram and exact sequences of left R-modules over a ring R:



State everything that you can conclude and prove about the maps: a, b, c.

(3) Let $f : V \to V$ be a nilpotent endomorphism of an n-dimensional vector space. Then $\operatorname{rank}(1 - f) = n$. Prove or give a counterexample.

(4) Let $f : V \to W$ be a linear transformation, where V and W are finite dimensional vector spaces over a field k. Prove that there exists a well defined linear transformation

$$f \wedge f : V \wedge V \to W \wedge W,$$

such that $(f \wedge f)(v \wedge v') = f(v) \wedge f(v')$ for all $v, v' \in V$.

(5) Let $f : V \to W$ be a linear transformation given by multiplication by the matrix:

$$\begin{bmatrix} 1 & 7 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}.$$

(a) Find a matrix for the transformation $f \wedge f : V \wedge V \to W \wedge W$.

(b) Find the dimensions (*only dimensions, not the matrix!*) of a matrix for the transformation $f \otimes f : V \otimes V \to W \otimes W$.

(6) Let p be an odd prime. Show that any group of order $2p^2$ is solvable, i.e. has an abelian tower whose last element is $\{e\}$.

(7) Describe all isomorphism classes of abelian groups of order $7(3^5)$.

(8) Consider the following short exact sequence of abelian groups:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/5\mathbb{Z} \longrightarrow 0,$$

where f is multiplication by 5, and π is the canonical projection. Apply $\mathrm{Hom}_{Ab}(\mathbb{Z}, \ )$ to this sequence and describe the sequence of abelian groups that you obtained.

(9) State any problem in linear algebra, multi-linear algebra, groups, rings, modules, categories ... that you particularly liked, and solve it.