Northeastern University
Department of Mathematics

# Qualifying Exam in Topology

## April 2005

*Do the following six problems. Give proofs or justifications for each statement you make. Draw pictures when needed. Be as* **clear** *and* **concise** *as possible. Show all your work.*

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1. Let $\mathbb{R}$ be the set of all real numbers, endowed with the usual topology, and let $\mathbb{R}_{\text{fc}}$ be the same set, endowed with the finite-complement topology. Let $f\colon \mathbb{R} \to \mathbb{R}_{\text{fc}}$ be the map given by $f(x) = x$.
   (a) Is $f$ continuous?
   (b) Is $f$ open?
   (c) Is $f$ a homeomorphism?

2. A topological space $X$ is said to be *locally compact* if, for every $x \in X$, there exists a compact subspace which contains a neighborhood of $x$.
   (a) Show that any set $X$ endowed with the discrete topology is locally compact.
   (b) Show that the space of rational numbers $\mathbb{Q}$ (with the subspace topology inherited from $\mathbb{R}$) is not locally compact.
   (c) Give an example of a locally compact space $X$ and a continuous map $f\colon X \to Y$ such that $f(X)$ is not locally compact.
   (d) Now assume $f\colon X \to Y$ is both continuous and open. Show that $f(X)$ is locally compact.

3. Let $Y = \{(z, w) \in \mathbb{C}^2 \mid z \neq w\}$. Let $X = Y/S_2$, where the symmetric group $S_2$ acts on $Y$ by permutating the coordinates. Let $p\colon Y \to X$ be the projection map.
   (a) Find the fundamental group of $Y$.
   (b) Find the fundamental group of $X$.
   (c) Determine the induced homomorphim $p_*\colon \pi_1(Y) \to \pi_1(X)$.

4. Let $X$ and $Y$ be path-connected spaces, with basepoints $x_0$ and $y_0$. Let $X \bigvee Y$ be the wedge of the two spaces at the respective basepoints.
   (a) Give a condition insuring that $\pi_1(X \bigvee Y) \cong \pi_1(X, x_0) * \pi_1(Y, y_0)$.
   (b) Find a pair of simply-connected spaces $X$ and $Y$ for which $X \bigvee Y$ is *not* simply-connected.

5. Let $\mathbb{RP}^2$ be the real projective plane.
   (a) Find all the connected covering spaces of $X = \mathbb{RP}^2 \bigvee \mathbb{RP}^2$.
   (b) Find all the connected covering spaces of the Klein bottle, $K = \mathbb{RP}^2 \sharp \mathbb{RP}^2$.

6. Let $f\colon \mathbb{RP}^2 \to S^1$ be a continuous map. Is $f$ necessarily null-homotopic?