NORTHEASTERN UNIVERSITY
DEPARTMENT OF MATHEMATICS

Qualifying Exam in Topology

January 2006

Do the following six problems. Give proofs or justifications for each statement you make. Draw pictures when needed. Be as **clear** *and* **concise** *as possible. Show all your work.*

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1. Let $f\colon X \to Y$ be a continuous surjection, and let $g\colon Y \to Z$ be a function so that $g \circ f$ is continuous. Show that:
   (a) If f is a closed map, then g is continuous.
   (b) If f is not a closed map, then g may fail to be continuous.

2. Let $X = [0,1]/(\frac{1}{4}, \frac{3}{4})$ be the quotient space of the unit interval, where the open interval $(\frac{1}{4}, \frac{3}{4})$ is identified to a single point. Show that:
   (a) X is connected.
   (b) X is compact.
   (c) X is not Hausdorff.

3. (a) Define the two notions: "homotopy between two maps" and "homotopy equivalence between two topological spaces."
   (b) Give an example of topological spaces X and Y that have the same homotopy type but are not homeomorphic.
   (c) Give an example of topological spaces X and Y that have isomorphic fundamental groups but are not homotopy equivalent.
   (d) Give an example of topological spaces X and Y that have isomorphic homology groups (in all degrees) but are not homotopy equivalent.

4. Let M be a compact, connected, orientable surface of genus 3 with 2 boundary circles.
   (a) Compute the fundamental group and the homology groups of M.
   (b) Suppose N is a compact, connected, orientable surface that is a covering of M. Show that N has an even number of boundary circles.

5. Let K be the 2-skeleton of the 4-simplex.
   (a) Write down the simplicial chain complex of K.
   (b) Compute all the homology groups of K.

6. Let $X = S^1 \bigvee \mathbb{RP}^2$ be the one-point union of a circle and a projective plane. Describe **all** 2-fold coverings $p\colon E \to X$, with E path-connected. In each case:
   (a) Sketch the cover, and identify the homotopy type of E in terms of familiar spaces.
   (b) Identify the corresponding subgroup $p_*(\pi_1(E, e_0)) \subset \pi_1(X, x_0)$.