

## Qualifying Exam in Topology

September 2009

Do the following five problems. Give proofs or justifications for each statement you make. Draw pictures when needed. Be as **clear** and **concise** as possible. Show all your work.

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- Let  $f: X \rightarrow Y$  be a continuous, surjective map from a space  $X$  to a connected space  $Y$ . Assume  $f^{-1}(y)$  is connected, for each  $y \in Y$ .
  - Show that if  $f$  is a quotient map, then  $X$  is connected.
  - Give an example to show that if  $f$  is not a quotient map, then  $X$  need not be connected.
- Let  $f: X \rightarrow Y$  be a continuous map from a space  $X$  to a Hausdorff space  $Y$ . Let  $C$  be a closed subspace of  $Y$ , and let  $U$  be an open neighborhood of  $f^{-1}(C)$  in  $X$ .
  - Show that if  $X$  is compact then there is an open neighborhood  $V$  of  $C$  in  $Y$  such that  $f^{-1}(V)$  is contained in  $U$ .
  - Give an example to show that if  $X$  is not compact, then there need not be such a neighborhood  $V$ .
- For an integer  $n \geq 1$ , let  $S^n$  be the  $n$ -sphere,  $\mathbb{R}P^n$  the  $n$ -dimensional projective space, and  $T^n$  the  $n$ -dimensional torus.
  - For which values of  $n$  does there exist a continuous map  $S^n \rightarrow S^1$  which is not homotopic to a constant?
  - For which values of  $n$  does there exist a continuous map  $\mathbb{R}P^n \rightarrow S^1$  which is not homotopic to a constant?
  - For which values of  $n$  does there exist a continuous map  $T^n \rightarrow S^1$  which is not homotopic to a constant?
- Let  $A$  be a 2-holed torus (i.e., a compact, connected, orientable surface of genus 2) with an open 2-disk removed, and let  $B$  be another copy of the 2-disk.

Let  $f: \partial B \rightarrow \partial A$  be the map winding three times (upon identifying  $\partial A$  and  $\partial B$  with the unit circle  $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ , the map  $f$  is given by  $f(z) = z^3$ ).

Finally, let  $X = A \cup_f B$  be the space obtained from  $A$  by adjoining the 2-cell  $B$  along the attaching map  $f$ .

  - Compute the fundamental group of  $X$ .
  - Compute all the (integral) homology groups of  $X$ .
- Let  $B = S^1 \vee S^1$  be the wedge of two circles. Find at least 4 non-equivalent 3-fold covering spaces  $p: E \rightarrow B$ , with  $E$  path-connected. In each case:
  - Draw a picture of the cover, clearly indicating how the projection map  $p$  works.
  - Compute the induced homomorphism on fundamental groups,  $p_{\#}: \pi_1(E, e) \rightarrow \pi_1(B, b)$ , for some conveniently chosen basepoints  $e$  and  $b$  with  $p(e) = b$ .
  - Compute the induced homomorphism on first homology groups,  $p_*: H_1(E, \mathbb{Z}) \rightarrow H_1(B, \mathbb{Z})$ .
  - Indicate whether the cover is regular or not.