NORTHEASTERN UNIVERSITY
DEPARTMENT OF MATHEMATICS

Qualifying Exam in Topology

May 2008

Do the following five problems. Give proofs or justifications for each statement you make. Draw pictures when needed. Be as **clear** *and* **concise** *as possible. Show all your work.*

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1. Let $(X, \mathcal{T})$ be a topological space. Show that the following are equivalent:
   (a) X is locally connected
   (b) The family of open connected subsets of X is a basis for $\mathcal{T}$.
   (c) For every $U \in \mathcal{T}$, each component of U is open in X.

2. (a) Define the two notions: "homotopy between two maps" and "homotopy equivalence between two topological spaces."
   (b) Give an example of topological spaces X and Y that have the same homotopy type but are not homeomorphic.
   (c) Give an example of topological spaces X and Y that have isomorphic fundamental groups but are not homotopy equivalent.
   (d) Give an example of topological spaces X and Y that have isomorphic first homology groups but non-isomorphic fundamental groups.

3. Consider two labeled dodecagons with edges identified in pairs, as follows:
$$abcdb^{-1}a^{-1}e^{-1}f^{-1}efc^{-1}d^{-1}, \qquad abcdb^{-1}a^{-1}e^{-1}f^{-1}efc^{-1}d,$$
   and let Σ_1 and Σ_2 be the two resulting surfaces.
   (a) Determine the orientability status, the genus, and the Euler characteristic of Σ_1 and Σ_2.
   (b) Find presentations for the fundamental groups of Σ_1 and Σ_2.
   (c) Compute the homology groups $H_i(\Sigma_1, \mathbb{Z})$ and $H_i(\Sigma_2, \mathbb{Z})$, for all $i \geq 0$.

4. Let K be the Klein bottle.
   (a) Identify (up to homeomorphism) all the path-connected spaces E that appear as the total space of a covering map $p\colon E \to K$. Which one of those is the universal cover? What are the isomorphism types of the corresponding subgroups?
   (b) Prove, or give a counterexample to the following assertion: Every continuous map $S^1 \to K$ is null-homotopic.
   (c) Prove, or give a counterexample to the following assertion: Every continuous map $S^2 \to K$ is null-homotopic.