Capacity and its applications

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Abstract

Capacity appears in electrostatics as a characteristic of a body made of conducting material: it is the maximal charge that can be put on this body so that the electric potential of the field created by this charge is bounded by 1. In 1924 Norbert Wiener introduced capacity in mathematics as a positive function of compact sets. It is not additive as a measure but subadditive. Starting with Wiener's famous work, the theory was fast developed and applied in many problems of analysis, partial differential equations, mathematical physics and geometry.

This paper presents a short summary of my lectures given in Summer 2004 for graduate students in Northeastern University (Boston) and in Conference-School on Analysis and Geometry (Novosibirsk). It starts with a definition and simplest properties of the Wiener capacity. Then we describe some classical applications of capacity in partial differential equations: removable singularities of bounded harmonic functions and regularity of boundary points for the Dirichlet boundary value problem (Wiener). In the last section we formulate a recent result by V.Maz'ya and M.Shubin on two-sided estimates for the bottom of the spectrum of the Laplacian with the Dirichlet boundary conditions in open subsets of $\mathbb{R}^n$. We comment on corollaries of this result and formulate unsolved problems.

1 Introduction to capacity

In this section we describe definition and simplest properties of the Wiener capacity. The detailed proofs and more details can be found e.g. in [2].

Capacity is a function on sets

$$\text{cap} : \{ \text{ Borel subsets of } (\mathbb{R}^n)\} \longrightarrow [0, +\infty]$$

where for simplicity we only consider the case $n \geq 3$.

We will only need compact sets $F \subset \mathbb{R}^n$. In this case $0 \leq \operatorname{cap}(F) < +\infty$. The notion of capacity comes from electrostatics, where a unit of capacity is called **Farad**, in memory of a great English scientist Michael Faraday(1791-1867). Faraday first was a bookbinder. Though being self-trained and having no grasp of mathematics, he became interested in electricity and eventually became a great physicist. He discovered electromagnetic induction and introduced the notion of field. His picture is printed on 20 British Pounds banknotes.

Capacity characterizes an electric device which is called *capacitor* and used to store electric energy by accumulating imbalance of electric charge. 1 coulomb of charge causes a 1 volt difference of potentials across 1 farad capacitor. We can formulate this as the relation

$$V = \frac{Q}{C}$$

where V is the voltage drop, Q is the charge of the body and C is the capacitance or capacity of the capacitor. V can also be considered as the work which is needed to drag a unit charge through the capacitor. Since farad is a very big unit (SI), generally we use smaller units: microfarad ΩF, nanofarad nF and picofarad pF which are equal to $10^{-6}F$, $10^{-9}F$, $10^{-12}F$ respectively.

It is known from electrostatics that a charge q at $x_0 \in \mathbb{R}^3$ creates electric field $\mathbb{E} = -\nabla V$ at any point $x \in \mathbb{R}^3$. Here

$$V(x) = \frac{q}{4\pi|x - x_0|},$$

hence by an easy calculation

$$\mathbb{E} = \frac{q}{4\pi} \frac{\vec{x} - \vec{x}_0}{|\vec{x} - \vec{x}_0|^3},$$

which corresponds to the inverse squares law of interaction discovered in electrostatics by Coulomb (and earlier in gravitation by Newton). Since the force acting on a charge e at x is $e\mathbb{E}(x)$, then the work done over a test charge e to drag it along a curve $\gamma : [0,1] \to \mathbb{R}^3$ against the field is

$$\int_\gamma (-e\mathbb{E}) \cdot d\vec{x} = \int_\gamma e\, dV = e(V(\gamma(1)) - V(\gamma(0))),$$

In particular, we see that this work is independent of the curve, provided that the initial and terminal points of the curve are fixed.

In n-dimensional case ($n \geq 3$) a natural generalization of the potential above is obtained if we take $V(x) = q\mathcal{E}(x)$ where

$$\mathcal{E}(x) = \frac{1}{(n-2)\omega_n|x - x_0|^{n-2}}, \quad x \in \mathbb{R}^n,$$

which is the fundamental solution for the operator $(-\Delta)$ in $\mathbb{R}^n$, $n \geq 3$. In this formula ω_n is the $(n-1)$-dimensional volume of the $(n-1)$-dimensional unit sphere.

If we have several charges $q_1, q_2, ..., q_N$ at the points $y_1, y_2, ..., y_N$ respectively, then we can use so called *superposition principle* which is equivalent to the linearity of the electrostatics equations (or, more generally, Maxwell equations, which describe electrodynamics): if charges are added, then the forces and potentials are added too. So we get

$$V(x) = \frac{1}{4\pi} \sum_{i=1}^{N} \frac{q_i}{|x - y_i|}, \quad x \in \mathbb{R}^3,$$

and more generally

$$V(x) = \sum_{i=1}^{N} q_i \mathcal{E}(x - y_i), \quad x \in \mathbb{R}^n, \ n \geq 3.$$

In case when the charge distribution is not discrete and given by a density function $\rho(x)$ (say, continuous and compactly supported), then we need to take the potential in the form

$$V(x) = \int_{\mathbb{R}^n} \mathcal{E}(x - y)\rho(y)dy,$$

or more generally if distribution is given by a compactly supported measure (possibly signed) μ:

$$V(x) = \int_{\mathbb{R}^n} \mathcal{E}(x - y)d\mu(y).$$

Note that $\Delta V(x) = 0$ outside the support of the measure μ. If $d\mu(x) = \rho(x)dx$, where $\rho \in C^1$, then $\Delta V(x) = -\rho(x)$.

Now let us turn to a precise definition of capacity which is due to Norbert Wiener(1894-1964). He contributed to many areas of mathematics and applied mathematics. (In particular, he is known as the father of cybernetics.) He once said: "One of the chief duties of the mathematician in acting as an advisor to scientists is to discourage them from expecting too much from mathematics". Wiener gave the following definition of capacity:

Definition 1.1 *Capacity* of a compact set $F \subset \mathbb{R}^n$, $n \geq 3$, is

$$\text{cap}\,(F) = \sup_{\mu}\{\mu(F) | \int_F \mathcal{E}(x - y)d\mu(y) \leq 1 \text{ for all } x \in \mathbb{R}^n \setminus F\}, \tag{1.1}$$

where μ is a measure on F (possibly signed), supremum is taken over all such measures.

In fact, maximum or maximizing measure on F exists and it is unique. It is positive and supported on the boundary of F, which is $\partial F = F \setminus \mathrm{Int}(F)$, where $\mathrm{Int}(F)$ is the set of all interior points of F. The maximizing measure is called *equilibrium distribution of charges*. When the total measure $\mu(F)$ is fixed, then the equilibrium distribution of charges minimizes the energy of the system of charges. The corresponding potential (the integral in (1.1)) is called the *equilibrium potential*.

It was Faraday who demonstrated that in equilibrium the charges only reside on the exterior boundary of a charged conductor, and an exterior charge had no influence on anything enclosed within a conductor (this shielding effect is used in what is now known as the Faraday cage). This means that $V(x)$ is constant on every connected component of $\mathrm{Int}(F)$ (so $V = const$ on $\mathrm{Int}(F)$ if $\mathrm{Int}(F)$ is connected). This property is equivalent to saying that $\mathbb{E}(x) = 0$ on $\mathrm{Int}(F)$.

Let us also provide an alternative definition of the Wiener capacity:

Definition 1.2 *Capacity* of a compact set $F \subset \mathbb{R}^n$, $n \geq 3$, is

$$\mathrm{cap}\,(F) = \inf_u \left\{ \int_{\mathbb{R}^n} |\nabla u|^2 dx \;\middle|\; u \equiv 1 \text{ on } F,\ u(x) \to 0 \text{ as } |x| \to \infty \right\}. \tag{1.2}$$

Here the infimum is taken over all $u \in C^\infty(\mathbb{R}^n)$ satisfying the conditions in (1.2). It is easy to see that instead we can take infimum over functions $u \in C_0^\infty(\mathbb{R}^n)$, such that $u = 1$ in a neighborhood of F (with the neighborhood depending upon u). In yet another convenient version instead of requiring $u \in C^\infty$ (or C_0^∞) we can take Lipschitz functions i.e. functions u satisfying

$$|u(x) - u(y)| \leq C|x - y|, \quad x, y \in \mathbb{R}^n,$$

with C depending upon u. Such functions are known to be differentiable almost everywhere with the derivatives coinciding with the corresponding distributional derivatives (so that we can integrate by parts).

Definition 1.2 is equivalent to Definition 1.1 but we will not prove this now. Let us only mention that the minimizing function in (1.2) in fact coincides with the equilibrium potential for any set F as in Definition 1.1, provided F is sufficiently regular (e.g. if every connected component of F is the closure of an open set with a smooth boundary).

Proposition 1.3 *The capacity as a function on compact sets with values in* $[0, +\infty)$ *has the following properties:*

1. *Monotonicity:* $F_1 \subset F_2$ *implies that* $\mathrm{cap}\,(F_1) \leq \mathrm{cap}\,(F_2)$.
2. *Continuity: for every compact* F *and every* $\epsilon > 0$, *there exists an open set* $U \supset F$, *such that for every compact* F' *with* $U \supset F' \supset F$, *we have*

$$\mathrm{cap}\,(F') \leq \mathrm{cap}\,(F) + \epsilon.$$

3. *Choquet inequality : for any compact sets* $F_1, F_2 \subset \mathbb{R}^n$

$$\mathrm{cap}\,(F_1 \cup F_2) + \mathrm{cap}\,(F_1 \cap F_2) \le \mathrm{cap}\,(F_1) + \mathrm{cap}\,(F_2).$$

In particular, the capacity is subadditive i.e.

$$\mathrm{cap}\,(F_1 \cup F_2) \le \mathrm{cap}\,(F_1) + \mathrm{cap}\,(F_2)$$

for any compact sets $F_1, F_2 \subset \mathbb{R}^n$.

Proof. Monotonicity follows immediately from Definition 1.2. Continuity easily follows from the same definition of $\mathrm{cap}\,(F)$ if we use the test functions from $C_0^\infty(\mathbb{R}^n)$ which are equal to 1 near F.

To prove the Choquet inequality we can use test functions $u,\ v \in C_0^\infty(\mathbb{R}^n)$ which have compact support and are almost minimizing for $F_1,\ F_2$ in (1.2), and then take $\varphi = \max\{u, v\}$, $\psi = \min\{u, v\}$ (which are Lipschitz functions with compact support). Then φ, ψ can be test functions for $F_1 \cup F_2$, $F_1 \cap F_2$, and

$$\mathrm{cap}\,(F_1 \cup F_2) \ + \ \mathrm{cap}\,(F_1 \cap F_2) \le \int |\nabla \varphi|^2 dx + \int |\nabla \psi|^2 dx \tag{1.3}$$

$$= \int |\nabla u|^2 dx + \int |\nabla v|^2 dx \le \mathrm{cap}\,(F_1) + \mathrm{cap}\,(F_2) + \epsilon,$$

where $\epsilon > 0$ can be made arbitrarily small.

In this calculation we used the fact (which follows from the implicit function theorem) that for any function $f \in C^\infty(\mathbb{R}^n)$ the set

$$\{x : \ f(x) = 0, \nabla f(x) \neq 0\}$$

has measure zero (we should apply this to $f = u - v$). The resulting inequality in (1.3) holds for every $\epsilon > 0$, hence for $\epsilon = 0$, which ends the proof. $\square$

It can be shown that the Choquet inequality allows to extend capacity to all Borel sets, like a measure. We can start with the following two functions of sets which are related to capacity and defined for any set in $\mathbb{R}^n$ (unlike capacity):

Definition 1.4

- *Internal capacity of any set* $E \subset \mathbb{R}^n$ *is defined as*

$$\underline{\mathrm{cap}}\,(E) = \sup_{K \subset E,\ K\ compact} \mathrm{cap}\,(K)$$

- *External capacity of any set* $E \subset \mathbb{R}^n$ *is defined as*

$$\overline{\mathrm{cap}}\,(E) = \inf_{G \supset E,\ G\ open} \underline{\mathrm{cap}}\,(G).$$

As in measure theory the best sets are the ones where the above two definitions coincide. There is enough of them due to the following Choquet theorem:

Theorem 1.5 $\underline{\text{cap}}(E) = \overline{\text{cap}}(E)$ *for any Borel set* E *(and even any analytic set).*

Let us recall that the Borel sets are sets from a minimal σ-algebra which contains all open (or closed) sets. Analytic sets form a bigger σ-algebra which we will not define here.

Any set E satisfying the above condition in Theorem 1.5 is called *capacitable.* In particular, all open and closed sets are capacitable, as well as all sets which are obtained from them by arbitrarily many countable unions and intersections.

The following theorem establishes a relation between Lebesgue measure and capacity:

Theorem 1.6 *For any Borel set* $F \subset \mathbb{R}^n$, $n \geq 3$,

$$\text{mes}\,(F) \;\leq c_n(\,\text{cap}\,(F))^{\frac{n}{n-2}},$$

with equality for any closed ball.

The constant c_n can be found from the explicit values of the measure and capacity of the unit ball.

Corollary 1.7 *If* $F \subset \mathbb{R}^n$ *and* $\text{cap}\,(F) = 0$, *then* $\text{mes}\,(F) = 0$.

Note that the converse is not always true. For example, for any open ball B_r with the radius r we have

$$\text{cap}\,(\bar{B}_r) = \text{cap}\,(\partial B_r) = (n-2)\omega_n r^{n-2},$$

but $\text{mes}\,(\partial B_r) = 0$. Here $\bar{B}_r$ means the closure of B_r, i.e. the corresponding closed ball.

2 Applications to Partial Differential Equations

We will describe without proofs two important applications of capacity in partial differential equations. In these applications the language of capacity is clearly relevant, in particular because they give necessary and sufficient conditions of some important properties to hold. For the proof we refer the reader to [2].

2.1 Removable singularity property

Definition 2.1 *Let* E *be a compact subset in* $\mathbb{R}^n$. *Suppose that for any open set* $\Omega \subset \mathbb{R}^n$, *such that* $\Omega \supset E$, *and any* $u \in C^\infty(\Omega \setminus E)$ *such that* $\Delta u = 0$ *in* $\Omega \setminus E$ *and* u *is bounded on* $\Omega \setminus E$, *there exists* $U \in C^\infty(\Omega)$, *such that* $\Delta U = 0$ *in* Ω *and* $U = u$ *on* $\Omega \setminus E$ *(i.e., any bounded harmonic function in* $\Omega \setminus E$ *can be extended to a harmonic function in* Ω*). Then* E *is said to have removable singularities property.*

It is proved in PDE textbooks that a set consisting of one point satisfies the removable singularity property.

The following theorem completely characterizes all compact sets in $\mathbb{R}^n$ which have the removable singularity property.

Theorem 2.2 *A compact set E has the removable singularity property if and only if* $\operatorname{cap}(E) = 0$ *.*

Note that if E is a submanifold then

$$\operatorname{cap}(E) = 0 \Longleftrightarrow \operatorname{codim} E \geq 2.$$

2.2 Solvability of the Dirichlet Problem

Consider the Dirichlet problem stated in the classical form as follows:

$$\Delta u = 0 \text{ in } \Omega \subset \mathbb{R}^n, \qquad u|_{\partial\Omega} = \varphi \in C(\partial\Omega), \tag{2.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $\partial\Omega$ its boundary, φ is a given continuous function on $\partial\Omega$ and we are looking for a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$. (Then u is called a *classical solution.*) By the maximum principle such solution is unique. If $\partial\Omega$ is smooth or piecewise smooth, then the classical solution exists for all φ. But for general Ω the solution u may not exist for some φ.

N.Wiener discovered a necessary and sufficient condition on Ω such that the classical solution exists for all φ. We will now describe this result which is called Wiener Criterion.

Let us start by presenting Ω in the form of a union of domains with smooth boundaries:

$$\Omega = \bigcup_{k=1}^{\infty} \Omega_k, \quad \Omega_k \subset \Omega_{k+1},$$

and assume that $\varphi = \Phi\,|_{\partial\Omega}$ where $\Phi \in C(\bar{\Omega})$, i.e. Φ is a continuous extension of φ to $\bar{\Omega}$. Let us construct a harmonic function u_k in Ω_k such that

$$u_k\,|_{\partial\Omega_k} = \Phi\,|_{\partial\Omega_k}$$

Clearly,

$$|u_k| \leq M = \max_{\bar{\Omega}} \Phi.$$

It follows that the set of all u_k's is precompact in $C(K)$ for any compact $K \subset \Omega$. Therefore, passing to a subsequence if necessary, we can assume that $u_k \to U \in C^\infty(\Omega)$ uniformly on any compact set $K \subset \Omega$.

It is easy to see that U does not depend on Φ and if (2.1) is solvable, then $u = U$. However, in general, it is not necessarily true for all points $x \in \partial\Omega$ that

$$\lim_{x \in \Omega, x \to \bar{x}} U(x) = \varphi(\bar{x}).$$

The points $\bar{x}$ where this is true for all φ are called *regular*. (We will also call a point *irregular* if it is not regular.) The Dirichlet problem (2.1) is solvable for all φ if and only if all points $x \in \partial\Omega$ are regular.

The following theorem gives a regularity criterion:

Theorem 2.3 (Wiener) *A point $\bar{x} \in \partial\Omega$ is regular if and only if*

$$\sum_{k=1}^{\infty} 2^{k(n-2)} \operatorname{cap}\left((\bar{B}_{2^{-k}} \setminus B_{2^{-k-1}}) \cap (\mathbb{R}^n \setminus \Omega)\right) = +\infty,$$

where $B_r = B_r(\bar{x})$ (the open ball of radius r centered at $\bar{x}$), $\bar{B}_r$ is the closure of this ball.

The following theorem asserts that there are sufficiently many regular points:

Theorem 2.4 (Kellogg) *For any bounded open set $\Omega \subset \mathbb{R}^n$*

$$\operatorname{cap}\{\textit{irregular points on } \partial\Omega\} = 0.$$

It can be shown that then $(n-1)$-dimensional Hausdorff measure of the set of irregular points is 0. In particular, the set of regular points is dense in $\partial\Omega$.

3 An application of capacity in spectral theory

In this section we describe a recent application of capacity to spectral theory, based on ideas of [1].

Let Ω be a bounded open set in $\mathbb{R}^n$ with a smooth (C^∞) boundary $\partial\Omega$. Consider the operator $-\Delta$ on the domain

$$D(-\Delta) = \{u \in C^2(\bar{\Omega}),\ u|_{\partial\Omega} = 0\}.$$

Now let λ be an eigenvalue of the operator $(-\Delta)$ with a corresponding eigenfunction u, i.e. $u \in D(-\Delta)$, $u \not\equiv 0$, and $(-\Delta)u = \lambda u$ in Ω. It is convenient to consider $-\Delta$ as an (unbounded) linear operator in the Hilbert space $L^2(\Omega)$ with the scalar product $(u, v) = \int_\Omega u\bar{v}dx$. Then for an eigenfunction u with the eigenvalue λ we obtain, integrating by parts (or using the Green formula):

$$\lambda(u,u) = (-\Delta u, u) = \int_\Omega |\nabla u|^2 dx = \int_\Omega \sum_{j=1}^{n} \left| \frac{\partial u}{\partial x_j} \right|^2 dx,$$

which implies

$$\lambda = \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx} \geq 0.$$

It can be proved that there exists a complete orthonormal system $\psi_1, \psi_2, ...$ of eigenfunctions of $-\Delta$ in $D(-\Delta)$, with the eigenvalues $\lambda_1, \lambda_2, \dots,$ so that

$-\Delta\psi_j = \lambda_j\psi_j$. The eigenvalues form a discrete set with the only accumulation point at $+\infty$, so we can enumerate them in the increasing order:

$$\lambda_1 \le \lambda_2 \le \lambda_3 \le \dots .$$

Here each eigenvalue is listed the number of times which is equal to its multiplicity. (In fact, it can be proved that the lowest eigenvalue is simple, so that $\lambda_1 < \lambda_2$.) Then

$$\min_j \lambda_j = \lambda_1 = \min_u \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}, \tag{3.1}$$

where the minimum is taken over all $u \in D(-\Delta)$. To show this note first that λ_1 is obviously greater or equal than the right hand side of (3.1) because we can take $u = \psi_1$. For the inverse inequality we should prove that the ratio in the right hand side of (3.1) is always greater or equal than λ_1 for any $u \in D(-\Delta)$. To this end let us expand u over the system $\{\psi_j\}$: $u = \sum_j c_j\psi_j$, where c_j are the corresponding Fourier coefficients. Then $(u, u) = \sum_j |c_j|^2$ and

$$(-\Delta u, u) = \sum_j \lambda_j |c_j|^2 \ge (\min_j \lambda_j) \sum_j |c_j|^2 = \lambda_1(u, u),$$

which proves the desired inequality.

It can be shown that instead of taking minimum over $u \in D(-\Delta)$ in (3.1) we can take infimum over all functions $u \in C_0^\infty(\Omega)$. (To prove this we need to approximate any function $u \in D(-\Delta)$ by functions from $C_0^\infty(\Omega)$, multiplying u by appropriate cut-off functions which vanish near the boundary, and then smoothing them to put them into $C_0^\infty(\Omega)$.) But then the corresponding infimum is well defined for any open set $\Omega \subset \mathbb{R}^n$ (possibly unbounded and having non-smooth boundary). So for any such Ω we can define its invariant

$$\lambda(\Omega) = \inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}. \tag{3.2}$$

In fact, $\lambda(\Omega)$ is also the bottom of the spectrum of a self-adjoint operator which can be obtained by the Friedrichs construction from the quadratic form which is the Dirichlet integral (the numerator in (3.2)). We will not describe the Friedrichs construction. The corresponding operator does not necessarily have discrete spectrum (or even eigenvalues) in $L^2(\Omega)$. It may have continuous spectrum like in the case of $\Omega = \mathbb{R}^n$. In general the spectrum may have a complicated nature instead, and investigating it is an important problem of spectral theory.

For general unbounded Ω it is important to know whether the spectrum is separated from 0, i.e. whether $\lambda(\Omega)$ is strictly positive. Clearly, positivity of $\lambda(\Omega)$ is equivalent to the inequality

$$\int_\Omega |\nabla u|^2 dx \ge \lambda \int_\Omega |u|^2 dx, \; u \in C_0^\infty(\Omega), \tag{3.3}$$

to hold with $\lambda > 0$ (independent of u), or in other words,

$$\int_{\Omega} |u|^2 dx \le C \int_{\Omega} |\nabla u|^2 dx, \quad u \in C_0^\infty(\Omega), \tag{3.4}$$

where $C = C(\Omega)$ is independent of u (the best possible C is $\lambda(\Omega)^{-1}$).

Example 3.1 *In 1 dimensional case, let $\Omega = \mathbb{R}$, and take $u = u_N \in C^\infty(\mathbb{R})$ such that $u_N = 1$ on $[-N, N]$, $u_N = 0$ on $[-\infty, -N-1) \cup (N+1, +\infty]$, $|u_N| \le 1$, and $|u_N'| \le C$ for some $C > 0$ (independent of N). Then*

$$\frac{\int |\nabla u_N|^2 dx}{\int |u_N|^2 dx} \le \frac{C^2}{N} \longrightarrow 0 \text{ as } N \longrightarrow \infty,$$

so the estimate (3.4) is impossible (hence $\lambda(\mathbb{R}) = 0$).

Example 3.2 *Now let us consider the case of a finite interval $\Omega = (0, \ell)$, where $\ell > 0$. Let $u \in C^1([0, \ell])$, $u(0) = 0$. Then we can write*

$$u(x) = \int_0^x u'(t) dt,$$

so by the Cauchy-Schwarz inequality

$$|u(x)|^2 \le x \int_0^x |u'(t)|^2 dt \le \ell \int_0^\ell |u'(t)|^2 dt,$$

hence integrating it over $(0, \ell)$, we obtain

$$\int_0^\ell |u|^2 dx \le \ell^2 \int_0^\ell |u'(t)|^2 dt.$$

This means that we can take $C = \ell^2$ in (3.4) and $\lambda = \ell^{-2}$ in (3.3), so we should have

$$\lambda((0, \ell)) \ge \ell^{-2}.$$

Note that in this case the eigenfunctions can be explicitly found: they are $\sin \frac{\pi k x}{\ell}$, $k = 1, 2, ...,$, and the eigenvalues are $\frac{\pi^2 k^2}{\ell^2}$. So the smallest eigenvalue for $(0, \ell)$ is in fact $\lambda((0, \ell)) = \pi^2/\ell^2$.

In this example we were able to obtain the estimate of the form (3.4) for $\Omega = (0, \ell)$ with the only condition $u(0) = 0$.

Now we will try to characterize compact sets $F \subset \bar{B}_r$ (here B_r means a fixed open ball with an arbitrary center in $\mathbb{R}^n$, $\bar{B}_r$ its closure), such that the following estimate holds

$$\int_{B_r} |u|^2 dx \le C \int_{B_r} |\nabla u|^2 dx, \quad u \in C^\infty(\bar{B}_r), u|_F = 0,$$

where C is independent of u. The following example shows that for the dimension $n \ge 3$ it is not enough to take F consisting of a single point.

Example 3.3 *Let $B_r = B_r(0),\ F = \{0\}$ in $\mathbb{R}^n, n \ge 3$. Let us show that the estimate*

$$\int_{B_r} |u|^2 dx \le C \int_{B_r} |\nabla u|^2 dx,\ u \in C^\infty(\bar{B}_r),\ u(0) = 0,$$

does not hold. To this end take $u = u_\epsilon = u_\epsilon(|x|)$, where $u_\epsilon = 1$ if $|x| \ge \epsilon$, $u_\epsilon(0) = 0$, $|\nabla u| \le C_1 \epsilon^{-1}$. Then $\int_{B_r} |u_\epsilon|^2 dx \ge C_2 > 0$, but

$$\int_{B_r} |\nabla u|^2 dx \le C_3 \epsilon^{-2} \epsilon^n = C_3 \epsilon^{n-2} \longrightarrow 0,\ \text{as } \epsilon \searrow 0,$$

which contradicts the estimate above.

In fact, it can be shown that the estimate

$$\int_{B_r} |u|^2 dx \le C \int_{B_r} |\nabla u|^2 dx,\ u \in C^\infty(\bar{B}_r),\ u|_F = 0,$$

holds if and only if $\operatorname{cap}(F) > 0$. More precisely,

$$\int_{B_r} |u|^2 dx \le \frac{C_n r^n}{\operatorname{cap}(F)} \int_{B_r} |\nabla u|^2 dx,\ u \in C^\infty(\bar{B}_r),\ u|_F = 0,$$

where C_n depends on the dimension n only.

Now let us try to estimate $\lambda(\Omega)$ for an arbitrary open set $\Omega \subset \mathbb{R}^n$ in geometric terms. To this end we will first notice that $\lambda(\Omega)$ has the following *monotonicity property*:

$$\text{If } \Omega' \subset \Omega, \text{ then } \lambda(\Omega) \subset \lambda(\Omega').$$

This immediately follows from (3.2) because the supply of functions for taking the infimum is larger and therefore the infimum is smaller for Ω compared with Ω'.

Now let us compare Ω with a ball $B_r \subset \Omega$. By a scaling (a similarity transformation of variables reducing the ball B_r to a unit ball B_1), we easily obtain that

$$\lambda(B_r) = c_n r^{-2},$$

where $c_n = \lambda(B_1)$. By monotonicity we get then:

$$\lambda(\Omega) \le c_n r^{-2}.$$

This estimate becomes stronger if r increases, so we should take a ball of the maximal radius to get the best estimate. Since the biggest ball $B_r \subset \Omega$ may not exist, we can define the *interior radius* of Ω by

$$r_\Omega = \sup\{r \mid \exists B_r \subset \Omega\}. \tag{3.5}$$

Clearly, $0 < r_\Omega \leq +\infty$ for any non-empty Ω (and it can be indeed $+\infty$ for unbounded Ω). From the previous arguments we easily conclude that

$$\lambda(\Omega) \leq C_n r_\Omega^{-2}. \tag{3.6}$$

This gives the best result which we can get from monotonicity. But it is still far from being precise. In particular, the opposite estimate is not true. This can be seen from the following example. Let

$$\Omega = \mathbb{R}^n \setminus \bigcup_{z \in \mathbb{Z}^n} \bar{B}_{r_z}(z),$$

where $r_z \to 0$ sufficiently fast as $|z| \to \infty$ (e.g. $r_z = 2^{-|z|}$ is sufficient). Obviously, $r_\Omega < \infty$, but it can be proved (e.g. from a more precise estimate given below) that $\lambda(\Omega) = 0$.

However by modifying the definition of r_Ω in (3.5), we can improve (3.6) and get a two-sided estimate for $\lambda(\Omega)$. This modification consists of ignoring sets of "small" capacity, which we will refer to as *negligible* sets. In fact the definition of negligibility includes a parameter γ, $0 < \gamma < 1$. We will call a compact set $F \subset \bar{B}_r$ *negligible* in $\bar{B}_r$, or, more precisely, γ-*negligible* in $\bar{B}_r$, if

$$\operatorname{cap}(F) \leq \gamma \operatorname{cap}(\bar{B}_r).$$

Now we can modify the definition of r_Ω by introducing a new quantity

$$r_{\Omega,\gamma} = \sup\{r \,|\, \exists B_r \subset \mathbb{R}^n,\ \bar{B}_r \setminus \Omega \text{ is } \gamma\text{-negligible in } B_r\}.$$

So here we allow not only balls $B_r \subset \Omega$ but balls which are in Ω up to a set of "small" capacity. (Note, however, that here "small" may mean set which is allowed to take 99% os capacity of $\bar{B}_r$, if $\gamma = 0.99$.)

The quantity $r_{\Omega,\gamma}$ is called *interior capacitary radius* of Ω.

We will now formulate a result which is essentially contained in [1] and gives the desired two-sided estimate for $\lambda(\Omega)$.

Theorem 3.4 *Let us fix $\gamma \in (0,1)$. Then there exist $c = c(\gamma, n) > 0$ and $C = C(\gamma, n) > 0$ such that*

$$c\, r_{\Omega,\gamma}^{-2} \leq \lambda(\Omega) \leq C\, r_{\Omega,\gamma}^{-2}. \tag{3.7}$$

Let us formulate some interesting corollaries of Theorem 3.4.

Corollary 3.5 $\lambda(\Omega) > 0$ *if and only if* $r_{\Omega,\gamma} < \infty$.

This corollary gives necessary and sufficient condition of strict positivity of the operator $-\Delta$ (with the Dirichlet boundary conditions) in Ω. (Here the operator should be understood as the Friedrichs extension from $C_0^\infty(\Omega)$.)

Since the condition $\lambda(\Omega) > 0$ does not contain γ, we immediately obtain

Corollary 3.6 *Conditions $r_{\Omega,\gamma} < \infty$, taken for different γ's, are equivalent.*

Denoting $F = \mathbb{R}^n \setminus \Omega$ (which can be an arbitrary closed subset in $\mathbb{R}^n$), we obtain from the previous Corollary (comparing $\gamma = 0.01$ and $\gamma = 0.99$):

Corollary 3.7 *Let F be a closed subset in $\mathbb{R}^n$, which has the following property: there exists $r > 0$ such that*

$$\operatorname{cap}(B_r \setminus \Omega) \geq 0.01 \operatorname{cap}(B_r)$$

for all B_r. Then there exists $r_1 > 0$ such that

$$\operatorname{cap}(B_{r_1} \setminus \Omega) \geq 0.99 \operatorname{cap}(B_{r_1})$$

for all B_{r_1}.

This is a new property of capacity which is proved by spectral theory arguments.

Let us formulate two open problems related with the topics discussed in this section.

1. Find precise dependence of $c = c(\gamma, D)$ and $C = C(\gamma, n)$ from (3.7) upon γ and n.

2. Extend the results formulated in this section to the Laplacian on Riemannian manifolds.

Once upon a time Marc Kac formulated a fundamental and fascinating question: **"Can you hear the shape of a drum?"** The precise meaning of this question is as follows: is it possible to reconstruct the drum (a bounded domain in $\mathbb{R}^2$) up to an isometry by the spectrum of its Dirichlet Laplacian (i.e. Laplacian with the Dirichlet boundary conditions)? Now, decades and hundreds (if not thousands) papers after this formulation first appeared, this question and its generalizations are still in the focus of attention for many researchers in spectral geometry.

Theorem 3.4 suggest formulation of a question, which is roughly inverse to the question of Marc Kac: **"Can you see the fundamental frequency of a drum?"** More precisely, can you find a simple visual image related to a domain in $\mathbb{R}^2$ (or $\mathbb{R}^n$), such that it allows to recover the lowest eigenvalue of the Dirichlet Laplacian in this domain? Assuming that our eye can filter out the sets of small capacity, a partial answer to this question is given by Theorem 3.4.

I will finish with a quote which I borrowed from a preface by Michel Hazewinkel to one of the books which he edited (and which is published by Kluwer Publishers): **"Approach your problems from the right end and begin with the answers. Then one day, perhaps you will find the final question."** (From "The Hermit Clad in Crane Feathers" in R. van Gulik's *"The Chinese Maze Murders".*)

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