INDEX THEORY

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Lecture Notes taken by ***Nilufer Koldan***

In this lecture I will describe one of the most significant achievements of the second half of the XX century – the Atiyah-Singer index theorem. I will also discuss some more recent developments in the area as well as some open problems.

1. The definition of the index

1.1. **The finite-dimensional case.** Let $A : V_1 \to V_2$ be a linear map between finite dimensional vector spaces. Define

$$\mathrm{Coker}(A) = V_2 / \operatorname{Im} A.$$

Then

$$\dim \operatorname{Ker}(A) - \dim \operatorname{Coker}(A) \;=\; \dim(V_1) - \dim(V_2). \tag{1.1}$$

Example: $A : V \to V \Rightarrow \dim(\operatorname{Ker}(A)) - \dim(\operatorname{Coker}(A)) = 0$

Exercise: Let $A(t)$ be a continuous family of matrices then $\forall t_0\, \exists \epsilon > 0 : \forall t, |t - t_0| < \epsilon$ such that

$$\dim \operatorname{Ker} A(t) \;\le\; \dim \operatorname{Ker} A(t_0).$$

Hint: If $\dim(\operatorname{Ker} A(t_0)) = 0$ then $\|A(t_0)v\| \ge \delta \|v\|$.

Example: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $A(t) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$

In other words, the dimension of the kernel is a *semi-continuous* function of t. Similarly, the dimension of the cokernel is a semi-continuous function. But the equation (1.1) implies that the difference of those two functions is constant.

1.2. **The infinite - dimensional case.** Let now V be an infinite dimensional vector space and let $A : V \to V$ be a linear operator. Then A is not necessarily invertible even if $\operatorname{Ker} A = \{0\}$.
Example: $V = L^2(\mathbb{R}),\ Af(x) = xf(x)$.

Definition 1. A linear operator $A : V_1 \to V_2$ is called *Fredholm* if $\dim(\operatorname{Ker}(A)) < \infty$ and $\dim(\operatorname{Coker}(A)) < \infty$.

The following definition was first suggested by Fritz Noether

Definition 2. The *index* $\operatorname{Ind} A$ of a Fredholm operator $A : V_1 \to V_2$ is defined by the formula

$$\operatorname{Ind} A \;:=\; \dim(\operatorname{Ker}(A)) \;-\; \dim(\operatorname{Coker}(A)) \tag{1.2}$$

This definition is interesting because of the following theorems, which show the "stability" of the index

Theorem 3. *Let $A(t)$ be a continuous family of Fredholm operators. Then*

$$\operatorname{Ind} A(t) \;=\; \text{const}\,.$$

Theorem 4. *Suppose A is a Fredholm operator and K is a compact operator. Then*

$$\operatorname{Ind}(A+K) \;=\; \operatorname{Ind} A.$$

1.3. **An example of application of the stability of the index.** From (1.1) we see that if $A : V \to V$ and $\dim V < \infty$ then $\operatorname{Ind} A = 0$. This is not so, in general, if $\dim V = \infty$.

Example: Consider the Hilbert space $H = l_2 = \{(x_1, x_2, \dots) | \sum |x_i|^2 < \infty\}$. Let $T : l_2 \to l_2$ be the operator defined by

$$T(x_1, x_2, \dots) = (x_2, x_3, \dots).$$

Then $\operatorname{Ker} T = \{(x_1, 0, 0, \dots)\}$, $\dim \operatorname{Ker} T = 1$, $\dim \operatorname{Coker} T = 0$. Therefore $\operatorname{Ind} T = 1$.

The above calculation can be used to prove the existence of non-trivial solutions of certain linear equations. In particular, we have the following

Proposition 5. *For every bounded operator B and every sufficiently small number ϵ the linear equation*

$$(T + \epsilon B)\, x \;=\; 0$$

has a non-trivial solution.

Proof. From theorem 3 we obtain $\operatorname{Ind}(T + \epsilon B) = \operatorname{Ind}(T) = 1$ if ϵ is sufficiently small. Hence,

$$\dim \operatorname{Ker}(T + \epsilon B) \;\geq\; \dim \operatorname{Ker}(T + \epsilon B) \;-\; \dim \operatorname{Coker}(T + \epsilon B) \;=\; 1.$$

□

The above proposition illustrate one of the most standard ways of application of the index in analysis: from the fact that the index is positive one concludes that the operator has a non-trivial kernel.

1.4. **The equivariant index.** The index, as it defined in Definition 2, is just an integer number. If there is a symmetry involved, one can define a finer invariant.

Let a compact group G act on the spaces V_1, V_2. i.e., for each $g \in G$ there are given linear operators

$$\pi_i(g) : \; V_i \;\longrightarrow\; V_i, \qquad i = 1, 2,$$

such that $\pi_i(g_1) \circ \pi_i(g_2) = \pi_i(g_1 g_2)$ for every $g_1, g_2 \in G$.

Suppose that the operator $A : V_1 \to V_2$ is G-*equivariant*, i.e.,

$$A \circ \pi_1(g) \;=\; \pi_2(g) \circ A, \qquad \text{for every} \quad g \in G.$$

Then $\operatorname{Ker} A$ is a G-invariant subspace of V_1 since if $x \in \operatorname{Ker} A$ then

$$A\pi_1(g)x \;=\; \pi_2(g)(Ax) \;=\; \pi_2(g) \cdot \overline{0} \;=\; \overline{0} \;\Rightarrow\; \pi_1(g)x \in \operatorname{Ker} A.$$

Definition 6. Suppose a group G acts on a linear space W. Then W is called an *irreducible representation of* G if W has no nontrivial G-invariant subspaces.

Recall that *all finite dimensional representations of a compact group are direct sums of irreducibles.* In particular, we can write

$$\operatorname{Ker} A \;=\; \sum m_V^+\, V, \qquad \operatorname{Coker} A \;=\; \sum m_V^-\, V, \tag{1.3}$$

where the sum runs over all irreducible representations of G (note, however, that since the dimensions of the kernel and the cokernel of A are finite, only finitely many of the numbers $m_V^\pm$ could be non-zero. Hence, both sums are actually finite).

The set of all finite dimensional representations of G forms a *semigroup* under the operation $\oplus$.

Definition 7. Consider the *Grothendieck group* of the semigroup of finite dimensional representations of G, i.e., the collection of formal differences $V_1 - V_2$ (where V_i are finite dimensional representations of G) subject to the equivalence relation [1]

$$W_1 - W_2 \sim V_1 - V_2 \;\Leftrightarrow\; V_1 + W_2 = W_1 + V_2$$

This group is called the *ring of characters* of G and is denoted by $R(G)$.

Definition 8. Let $A : V_1 \to V_2$ be a G-equivariant linear operator. Using the notation introduced in (1.3), we define the equivariant index $\operatorname{Ind}_G(A)$ of A by the formula

$$\operatorname{Ind}_G(A) \;:=\; \operatorname{Ker} A \;-\; \operatorname{Coker} A \;=\; \sum (m_V^+ - m_V^-)\, V \;\in\; R(G). \tag{1.4}$$

Remark 9. We can consider infinite formal sums $\sum m_V V$. Then we obtain the *completed ring of characters* of G:

$$\hat{R}(G) \;=\; \Big\{ \sum m_V\, V : V \text{is an irr. repr. of } G,\ m_V \in G \Big\}.$$

2. The index of differential operators

2.1. **Differential Operators.** A differential operator of order k on $\mathbb{R}^n$ is

$$\mathcal{D} \;=\; \sum_{\sum \alpha_i \le k} a_{\alpha_1 \dots \alpha_n}(x) \frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}. \tag{2.1}$$

Example: $\mathcal{D} = \sum a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum b_l \frac{\partial}{\partial x_l} + c$, where $\operatorname{Ord} \mathcal{D} = 2$.

Definition 10. For every integer $k \ge 0$ we define the *Sobolev space* $H^k(\mathbb{R}^n)$ by

$$H^k(\mathbb{R}^n) \;=\; \{\, f : \mathbb{R}^n \to \mathbb{C} \mid \frac{\partial^\alpha f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \in L^2(\mathbb{R}^n),\ \alpha_1 + \cdots + \alpha_n \le k \,\}.$$

In other words, the Sobolev space $H^k(\mathbb{R}^n)$ consists of the square-integrable functions whose all partial derivatives of order $\le k$ are square-integrable.

Using local coordinate charts one can use (2.1) to define the notion of a differential operator on a manifold M. Similarily one can define a notion of the Sobolev space $H^k(M)$.

[1] In general, the definition of the equivalence relation in the Grothendieck group is slightly more complicated. But, in our case, it is equivalent to the one given here.

The space $H^k(M)$ can be endowed with a scalar product

$$\langle f,g\rangle_k \;:=\; \sum_{\alpha_1+\cdots\alpha_n\le k}\int_M \frac{\partial^{\alpha_1+\ldots+\alpha_n}f}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}\cdot\frac{\partial^{\alpha_1+\ldots+\alpha_n}\overline{g}}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}\,dx. \tag{2.2}$$

(In particular, $\langle\cdot,\cdot\rangle_0$ is just the usual L^2-scalar product). The space $H^k(M)$ with the above scalar product is a Hilbert space.

Suppose that M is a compact manifold. From (2.2) we immediately see that *a differential operator* $\mathcal{D}: H^k(M)\to L^2(M)$ *is bounded* if $\operatorname{Ord}\mathcal{D}\le k$. Moreover, we have the following

Theorem 11. *If* $\operatorname{Ord}\mathcal{D}<k$ *and* M *is compact then the operator* $\mathcal{D}: H^k(M)\to L^2(M)$ *is compact.*

From theorem 11 and the stability of the index we obtain the following

Corollary 12. *The index of* $\mathcal{D}$ *depends only on the derivatives of order* k*, i.e., on the term*

$$\sum_{\alpha_1+\cdots+\alpha_n=k} a_{\alpha_1,\ldots,\alpha_n}\,\frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}. \tag{2.3}$$

2.2. **The principal symbol.** Our next goal is to describe the expression (2.3) in more invariant (i.e., coordinate free) terms.

Example: Consider the case when $\operatorname{Ord}\mathcal{D}=1$. Then (2.3) takes the form $\sum a_i\frac{\partial}{\partial x_i}$. When we apply this differential operator to the function f we obtain $\sum a_i\frac{\partial f}{\partial x_i}$, i.e., the *directional derivative of* f *along the vector field* $a(x)=(a_1(x),\ldots,a_n(x))$.

Let TM denote the tangent bundle to M. Let T^*M be the cotangent bundle to M, i.e, the bundle over M, whose fiber T^*_xM over a point $x\in M$ consists of the space of linear functions on the tangent space T_xM. Then the vector field $a(x)$ can be viewed as a *fiberwise liner map* $a: T^*M\to\mathbb{C}$. We conclude that, in this case, the expression (2.3) is defined by a fiberwise linear map $T^*M\to\mathbb{C}$.

In general, for an operator $\mathcal{D}$ of order k, the expression (2.3) defines a map $\sigma(\mathcal{D}): T^*_xM\to\mathbb{C}$ which is polynomial of order k. This map $\sigma(\mathcal{D})$ is called the *principal symbol* of D. It is very easy to express $\sigma(\mathcal{D})$ in local coordinates: Let $y_1,\cdots,y_n$ be local coordinates on M near a point $x\in M$. They define coordinates $\xi_1,\ldots,\xi_n$ on the linear space T^*_xM. Then

$$\sigma(\mathcal{D})(x,\xi) \;=\; \sum_{|\alpha|=k} a_{\alpha_1,\ldots,\alpha_n}\,(i\xi_1)^{\alpha_1}\ldots(i\xi_n)^{\alpha_n}.$$

Definition 13. $\mathcal{D}$ is called *elliptic* if $\sigma(\mathcal{D})(\xi)$ is invertible for $\forall\xi\ne0$.

Example: On $\mathbb{R}^2$, the *Laplace operator* $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is elliptic. Its leading symbol $\sigma(\Delta)$ is given by the formula

$$\sigma(\Delta)(\xi_1,\xi_2) \;=\; -\xi_1^2-\xi_2^2.$$

Theorem 14. *Every elliptic operator* $\mathcal{D}$ *on a compact manifold* M *is Fredholm.*

3. The Atiyah-Singer index theorem

As follows from corollary 12, the index of elliptic operator $\mathcal{D}$ depends only on the leading symbol $\sigma(\mathcal{D})$.

3.1. **The problem.** About 50 years ago Israel Gel'fand [10] formulated a problem: how to calculate the index of an elliptic operator using only its leading symbol. This problem was solved brilliantly by Atiyah and Singer [6, 7] (see also, for example, [11, 8]). My next goal is to describe their solution.

3.2. **A slight generalization.** One can consider also differential operators acting on vector-valued functions. More precisely, we consider two vector bundles E and F over M and our differential operator $\mathcal{D}$ acts from the space of sections of E to the space of sections of F:

$$\mathcal{D}:\, H^k(M,E) \ \to\ L^2(M,F).$$

Then the coefficients $a_{\alpha_1\dots\alpha_n}$ in (2.1) are matrices, or, more precisely, elements of $\mathrm{Hom}(E,F)$. In particular, the leading symbol is

$$\sigma(\mathcal{D}):\ T^*M \ \to\ \mathrm{Hom}(E,F),$$

or, in more details,

$$\sigma(\mathcal{D})(x,\xi):\, E_x \ \longrightarrow\ F_x.$$

3.3. **The Atiyah-Singer index theorem.** Atiyah and Singer did, in fact, more than just a calculation of the index. They associated to each elliptic symbol an element of the K-theory. Then they associated a number – *the topological index* – to each element of the K-theory. Schematically, their construction can be expressed as

$$\text{elliptic operator } \mathcal{D} \ \rightsquigarrow\ \sigma(\mathcal{D}) \ \rightsquigarrow\ \text{an element of } K\text{-theory} \ \rightsquigarrow\ \text{the topological index } \in \mathbb{Z}.$$

The composition of the arrows in the above diagram leads to a map

$$\text{elliptic operators} \ \to\ \mathbb{Z}, \qquad \mathcal{D} \mapsto\ \text{t-Ind}(\sigma(\mathcal{D})).$$

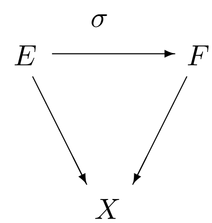
which is called the topological index. Note, that, as I will explain below, the topological index is constructed using purely topological method, without any analysis involved.

The following result is the simplest form of the Atiyah-Singer index theorem

Theorem 15. $\mathrm{Ind}\,\mathcal{D} \ =\ \text{t-Ind}(\sigma(\mathcal{D}))$

Remark 16. Though after the discussion in Section 2.1 it should be clear that the index of $\mathcal{D}$ could be calculated out of the principal symbol $\sigma(\mathcal{D})$, I call you to appreciate the power of the above theorem. The index of $\mathcal{D}$ gives you an information about the kernel and the cokernel of $\mathcal{D}$, i.e., about the spaces of solutions of differential equations $\mathcal{D}f = 0$ and $\mathcal{D}^*u = 0$. The index theorem allows to obtain this information *without solving the differential equations.*

3.4. ***K*-theory.** I will now explain in more details the construction of the topological index. First, I will briefly explain the notion of the (topological) K-theory. If X is a compact manifold then the K-theory of X is just the Grothendieck group of the semi-group of all vector bundles over X. In general the definition is a little bit more complicated. This is, essentially, because we are interested in the K-theory *with compact support.* In fact, one of the definitions of the K-theory of a non-compact manifold X is the K-theory of the *one-point compactification of* X. It will be more convenient for me to use an equivalent definition, which is based on considering bundle maps



such that the induced map of fibers $\sigma(x) : E_x \to F_x$ is invertible for all x outside of a compact set. Two such maps $\sigma_1 : E_1 \to F_1$ and $\sigma_2 : E_2 \to F_2$ are said to be equivalent if there exist integers $k_1, k_2 \geq 0$ such that the maps

$$\sigma_1 \oplus \mathrm{id} : E_1 \oplus \mathbb{C}^{k_1} \ \to \ F_1 \oplus \mathbb{C}^{k_1}, \quad \text{and} \quad \sigma_2 \oplus \mathrm{id} : E_2 \oplus \mathbb{C}^{k_2} \ \to \ F_2 \oplus \mathbb{C}^{k_2},$$

are *homotopic in the class of maps invertible outside of a compact set*. In this case we write $\sigma_1 \sim \sigma_2$.

Definition 17. Let X be a topological space. The *K-theory* $K(X)$ of X is defined by

$$K(X) \ = \ \{\sigma : E \to F \mid \sigma(x) \text{ is invertible outside of a compact set}\}/\sim$$

Remark 18. If X is compact then any two maps $\sigma_1, \sigma_2 : E \to F$ between the same bundles are equivalent. Thus $K(X)$ can be described as the set of pairs of vector bundles (E, F) subject to an appropriate equivalence relation. We think about this pair as about *formal difference* of the bundles E and F, and we, usually, denote this pair by $E - F$.

Exercise:

(1) Find the equivalence between the pairs (E_1, F_1) and (E_2, F_2) in the K-theory of a compact manifold.
(2) Give an example of equivalent pairs (E_1, F_1) and (E_2, F_2) such that the bundles $E_1 \oplus F_2$ and $E_2 \oplus F_1$ are not isomorphic.

Example: If $X = \{pt\}$ then a vector bundle over M is just a vector space and an element of $K(X)$ is a pair of two vector spaces (E, F) up to an equivalence. Using the above exercise, one readily sees that the only invariant of the pair (E, F) is the number

$$\dim E \ - \ \dim F \ \in \ \mathbb{Z}.$$

Thus

$$K(\{pt\}) \ \cong \ \mathbb{Z}. \tag{3.1}$$

Example: Let M be a manifold and let $\pi : T^*M \to M$ be its cotangent bundle. Let E and F be vector bundles over M. We denote by π^*E, π^*F the pull-backs of these bundles to T^*M.

Let $\sigma(\mathcal{D}) : E \to F$ be an elliptic symbol. Then for each $x \in M$ and each $\xi \in T_x^*M$ we have $\sigma(\mathcal{D})(x, \xi) : E_x \ \to \ F_x$. In other words, $\sigma(\mathcal{D})$ defines a bundle map

$$\sigma(\mathcal{D}) : \pi^\star E \ \to \ \pi^\star F. \tag{3.2}$$

If M is a compact manifold, then the ellipticity condition implies that the map (3.2) is invertible outside of a compact subset of $X = T^*M$. Hence, (3.2) defines an element of $K(T^*M)$.

One of the most important facts about the K-theory is that it has all the property of cohomology theory with compact supports. In particular, given an inclusion $j : Y \hookrightarrow X$ one can define a *push-forward map* $j_! : K(Y) \to K(X)$.

3.5. **The topological index.** Let now M be a compact manifold. Consider an embedding $M \hookrightarrow \mathbb{R}^N$ (such an embedding always exists for large enough N, by the Whitney embedding theorem). This embedding induces an embedding

$$j : T^*M \hookrightarrow \mathbb{R}^N \oplus \mathbb{R}^N \simeq \mathbb{C}^N.$$

We will denote by $i : \{pt\} \hookrightarrow \mathbb{C}^N$ an embedding of a point into $\mathbb{C}^N$. Using the push-forward in K-theory introduced above we obtain the diagram

$$\begin{array}{ccc} K(T^*M) & \xrightarrow{\;j_!\;} & K(\mathbb{C}^n) \\ & & \Big\uparrow i_! \\ & & K(\{pt\}) \end{array}$$

It is very important and non-trivial result, called *the Bott periodicity* states that the map $i_!$ is an isomorphism. Hence, we can invert it and define the *topological index* as the map

$$\text{t-Ind} = i_!^{-1} j_! : K(T^*M) \to K(\{pt\}) \simeq \mathbb{Z}. \tag{3.3}$$

It is relatively easy to check that this map is independent of the choice of the number N and the embedding $j : M \hookrightarrow \mathbb{R}^N$.

3.6. **The equivariant case.** If a compact group G acts on M one can define the *equivariant K-theory* $K_G(X)$ as the set of the equivalence classes of G-equivariant maps between G-equivariant vector bundles. In particular, G-equivariant K-theory of a point is given by equivalence classes of pairs of finite dimensional representations of G. It should not be a surprise that

$$K_G(\{pt\}) = R(G), \tag{3.4}$$

where $R(G)$ is the ring of characters of G, cf. definition 7. The formula (3.3) generalizes easily to define a G-equivariant topological index

$$\text{t-Ind}_G : K_G(M) \to K_G(\{pt\}) \simeq R(G).$$

All the constructions introduced above readily generalize to the equivariant setting. The index theorem of Atiyah and Singer is the following

Theorem 19. $\text{Ind}_G \mathcal{D} = \text{t-Ind}_G(\sigma(\mathcal{D}))$.

4. Current developments and open questions

There are enormously many directions in which the index theory was generalized. To name just a few of them I can mention the index theory of transversely elliptic operators [1], the index theory for manifolds with boundary [3, 4, 5], the index theory for covering manifolds [2], etc. There are also a lot of interesting and important applications of different index theorems and a lot of open questions. Here I will only very briefly mention one of the questions on which I was working recently.

An Open Question: If M is not compact and $\sigma : \pi^\star E \to \pi^\star F$ is a map invertible outside of a compact set, then the topological index $\text{t-Ind}(\sigma)$ is still defined. *What is the analytic analogue?*

This question is probably too general too be addressed. But some special case of this question was solved in [9], which led to important applications to geometric quantization and representation theory of non-compact groups. In particular, it allowed to construct a geometric quantization of a non-compact symplectic manifold. It also gave a new tool for proving different theorems about the index on compact manifolds.

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