READING COURSE ON SEMI–SIMPLE LIE ALGEBRAS FALL 2007

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1. Course description

This reading course will focus on the invariant theory of semi–simple Lie algebras and reflection groups. It can be taken in preparation for the graduate course on affine Lie algebras and integrable systems which I will give in Spring 2008, but is independent of it. It will assume some basic knowledge of semi–simple Lie algebras and root systems, as can be found for example in [Hu2]. However, since that material will only be treated from October 1, there will be ample time to do some background reading in preparation for it for those wishing to do so.

We will begin by reviewing Chevalley's theorems which generalise to any complex, semi–simple Lie algebra $\mathfrak{g}$ with Weyl group W the following well–known results, valid for $W = \mathfrak{S}_n$ and $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ respectively:

(1) The description of the ring $\mathbb{C}[\lambda_1, \ldots, \lambda_n]^{\mathfrak{S}_n}$ of symmetric functions in n variables as freely generated by the elementary symmetric functions

$$\sigma_k(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^{n} \lambda_i^k, \qquad k = 1, \ldots, n$$

(2) The isomorphism

$$\mathbb{C}[x_{ij}]^{GL_n} \cong \mathbb{C}[\lambda_1, \ldots, \lambda_n]^{\mathfrak{S}_n}$$

of the ring of invariant polynomial functions of an $n \times n$ matrix $X = (x_{ij})$ with that of symmetric functions in the eigenvalues $\lambda_1, \ldots, \lambda_n$ of X and, as a consequence, the free generation of the former ring by the trace functions $X \to \mathrm{tr}(X^k)$, $k = 1, \ldots, n$.

From a geometric point of view, these results imply that the quotients $\mathbb{C}/\mathfrak{S}_n$ and $\mathfrak{gl}_n(\mathbb{C})/GL_n(\mathbb{C})$ are affine spaces of dimension n.

We will then study Kostant's paper on three–dimensional subalgebras of semi–simple Lie algebras [Ko1] which gives, among many other things, a very elegant way of computating the Betti numbers, or exponents, of any semi–simple Lie group.

We will also study Kostant's slice theorem [Ko2] which gives a generalisation to all semi–simple Lie algebras of the familiar fact that a generic $n \times n$ matrix can be put in companion form:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & u_1 \\ 1 & 0 & \cdots & 0 & u_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & 0 & u_{n-1} \\ 0 & 0 & \cdots & 1 & u_n \end{pmatrix}$$

Time permitting, we will also study an infinite–dimensional analogue of Kostant's slice theorem due to Drinfeld and Sokolov which arises in the theory of integrable systems [DS]. In this example, the Lie algebra $\mathfrak{g}$ is replaced by an infinite–dimensional space of differential operators called Opers and the slice theorem gives a rational canonical form for these operators.

Printed copies of [Fr] will be distributed. The references [Cv, DS, Hu1, Ko1, Ko2] can be consulted on my website:
www.math.neu.edu/toledano/graduate_course_bibliography/

References

[Bo] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6.* Elements of Mathematics. Springer–Verlag, Berlin, 2002.

[Cv] C. Chevalley, *Invariants of finite groups generated by reflections*, Amer. J. Math. **77** (1955), 778–782.

[DS] V. G. Drinfeld, V. V. Sokolov, *Lie algebras and equations of Korteweg-de Vries type*, J. Sov. Math. **30** (1985) 1975–2035.

[Di] J. Dixmier, *Enveloping algebras*, Graduate Studies in Mathematics, 11. AMS, Providence, 1996.

[Fr] E. Frenkel, *Langlands Correspondence for Loop Groups. An Introduction.* Cambridge University Press, Cambridge, 2007.

[Hu1] J. E. Humphreys, *Reflection groups and Coxeter groups.* Cambridge Studies in Advanced Mathematics, 29. Cambridge University Press, Cambridge, 1990.

[Hu2] J. E. Humphreys, *Introduction to Lie algebras and representation theory.* Second printing, revised. Graduate Texts in Mathematics, 9. Springer–Verlag, New York-Berlin, 1978.

[Ko1] B. Kostant, *The principal three–dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math. **81** (1959) 973–1032.

[Ko2] B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **85** 1963, 327–402.

2. Organisation of the course

The reading course will be run as a seminar given by graduate students. We will meet every other Monday, starting on September 17, with the exceptions of October 15 and November 12 (Veterans' day), which will be replaced by Friday, October 12 and November 9 respectively, from 10:30 to 12:45 in room 509 LA. On October 12, the class will meet from 10:30 to 12:45 in room 544 NI. Each meeting will be divided into two hours (10:30–11:30 and 11:45–12:45) separated by a 10–15 minute break during. At each meeting one, or two, graduate student(s) will talk about a set subject. A tentative schedule follows.

Week	**Dates**	**Topic**	**Speaker**
1 **Sept. 3–9**	*Sept. 3: Labour day*		
2 **Sept. 10–16**			
3 **Sept. 17–23**	Sept. 17 (I) Sept. 17 (II)	Invariant theory for reflection groups [Bo, §V.5–V.6],[Cv],[Hu1, chap. 3]	J. Brown J. Brown
4 **Sept. 24–30**			
5 **Oct. 1–7**	Oct. 1 (I) Oct. 1 (II)	Invariant theory for semi–simple Lie algebras [Di, §7.4],[Hu2, chap. 23] Harish–Chandra's theorem [Di, §7.3],[Hu2, chap. 23]	D. Labardini D. Labardini
6 **Oct. 8–14**	(**Friday, 544NI**) Oct. 12 (I) (**Friday, 544NI**) Oct. 12 (II)	Kostant's TDS paper (I) [Ko1] Kostant's TDS paper (I) [Ko1]	S.–W. Yang S.–W. Yang
7 **Oct. 15–21**			
8 **Oct. 22–28**			
9 **Oct. 29–Nov. 4**	Oct. 29 (I) Oct. 29 (II)	Kostant's TDS paper (II) [Ko1] Kostant's slice paper (I) [Ko2, §4]	S.–W. Yang S. Gautam
10 **Nov. 5–11**	(**Friday**) Nov. 9 (II) (**Friday**) Nov. 9 (II)	Kostant's slice paper (II) [Ko2, §4] Kostant's slice paper (II) [Ko2, §4]	S. Gautam S. Gautam
11 **Nov. 12–18**	*Nov. 12: Veterans' day*		
12 **Nov. 19–25**	*Nov. 21–25: Thanksgiving*		
13 **Nov. 26–Dec. 2**	Nov. 26 (I) Nov. 26 (II)	Opers [DS, §3] [Fr, chap. 4] Opers [DS, §3] [Fr, chap. 4]	TBA TBA
14 **Dec. 3–9**			