READING COURSE ON POISSON LIE GROUPS NORTHEASTERN UNIVERSITY, SPRING 2008

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1. Course description

A *Poisson Lie group* is a Lie group G endowed with a Poisson structure such that the multiplication $G \times G \to G$ is a Poisson map. Poisson Lie groups were introduced by Drinfeld and Semenov–Tian–Shanskii in the mid–80's to resolve an important paradox in the study of integrable systems: in a number of cases of interest the phase space M possesses a Lie group G of 'hidden' symmetries which, however, do not preserve the Poisson structure on M. The explanation of this phenomenon was understood to lie in the fact that the group G itself possesses a non–trivial Poisson structure such that the action map $G \times M \to M$ is Poisson. In such situations, the quantum mechanical system associated to M is not symmetric under G but rather under a deformation of G, or *quantum group*, consistent with the Poisson structure.

We will begin by going over the definition of Poisson manifolds and Poisson Lie groups and prove an analogue of Lie's third theorem in this context which gives an equivalence between (connected and simply connected) Poisson Lie groups and *Lie bialgebras*. The latter are Lie algebras $\mathfrak{g}$ endowed with a Lie cobracket, that is a map $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ satisfying the co–Jacobi identity and a natural compatibility relation with the bracket on $\mathfrak{g}$.

We will then derive some interesting consequences of this equivalence, most importantly the existence of a *dual* G^* to a Poisson Lie group. When G is endowed with the trivial Poisson structure for example, G^* is the additive group underlying the vector space $\mathfrak{g}^*$ endowed with the Kostant–Kirillov Poisson structure.

We will continue to study *quasitriangular* bialgebras. These are bialgebras $\mathfrak{g}$ endowed with an element $r \in \mathfrak{g}\otimes\mathfrak{g}$ called a *classical r–matrix* such that the cobracket is given by

$$\delta(x) = (\mathrm{ad}(x) \otimes 1 + 1 \otimes \mathrm{ad}(x))\,(r)$$

The co–Jacobi identity implies that r has to satisfy the *classical Yang–Baxter equation* (CYBE)

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \in (\mathfrak{g}^{\otimes 3})^{\mathfrak{g}}$$

We will then cover Semenov–Tian–Shansky's theorem which states that if $\mathfrak{g}$ is a quasitriangular bialgebra endowed with an invariant inner product, the dynamical system associated with an invariant function $f \in C^\infty(\mathfrak{g})^{\mathfrak{g}}$ may be written in *Lax form*:

$$\frac{dL}{dt} = [L, B]$$

where L takes values in $End(\mathfrak{g})$. A direct consequence of this is that the symmetric functions of the eigenvalues of L, namely the functions $\mathrm{tr}(L^k)$, are in involution and therefore conserved under the corresponding flow.

Moreover, if r is *factorisable*, that is $\mathfrak{g}$ may be written as the orthogonal direct sum of two subalgebras $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ in such a way that $r \in \mathfrak{g} \otimes \mathfrak{g} \cong End(\mathfrak{g})$ is given by

$$r = P_+ - P_-$$

where $P_\pm : \mathfrak{g} \to \mathfrak{g}_\pm$ are the corresponding projections, the corresponding equations of motions can be explicitly solved.

Depending upon time, and the audience's wishes, we may look in more detail at a few examples of this procedure: the Toda lattice and the KdV equations.

Finally, we will study Drinfeld and Belavin's classification of solutions of the CYBE without, and with spectral parameters. In the latter case, these are CYBE where the finite–dimensional Lie algebra $\mathfrak{g}$ is replaced by the infinite–dimensional Lie algebra $\mathfrak{g}[u]$ of polynomial maps of $\mathbb{C}$ to $\mathfrak{g}$. The solutions $r = r(u)$ are then classified as *rational, trigonometric* and *elliptic* depending upon the rank of the lattice of poles of r.

References

[BD] A. A. Belavin, V. G. Drinfeld, *Triangle equations and simple Lie algebras.* Classic Reviews in Mathematics and Mathematical Physics, 1. Harwood Academic Publishers, 1998.

[CP] V. Chari, A. Pressley, *A guide to quantum groups.* Cambridge University Press, 1995.

[Dr] V. G. Drinfeld, *Quantum groups*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 798–820, Amer. Math. Soc., 1987.

[ES] P. Etingof, O. Schiffmann, *Lectures on quantum groups*, Lectures in Mathematical Physics. International Press, 2002.

[KS] L. I. Korogodski, Y. S. Soibelman, *Algebras of functions on quantum groups. Part I.* Mathematical Surveys and Monographs, 56. American Mathematical Society, Providence, 1998.

[Ko] Y. Kosmann–Schwarzbach, *Lie bialgebras, Poisson Lie groups and dressing transformations*, Integrability of nonlinear systems (Pondicherry, 1996), 104–170, Lecture Notes in Phys., 495, Springer, 1997.

[STS] M. A. Semenov–Tian–Shanskii, *What is a classical r–matrix?* Functional Anal. Appl. **17** (1983), 259–272.

[STS2] M. A. Semenov–Tian–Shanskii, *Dressing transformations and Poisson group actions*, Publ. Res. Inst. Math. Sci. **21** (1985), 1237–1260.

2. Organisation of the course

The reading course will be run as a seminar given by graduate students. We will meet every other Friday, starting on January 25, from 3:30 to 6:30 (with a break!) in room 511 LA.

Week	Dates	Lectures	Speaker
1 **Jan. 7–13**			
2 **Jan. 14–20**			
3 **Jan. 21–27**	*Jan. 21* Jan. 25	*Martin Luther King day, no classes* **Lecture 1**	 V. Toledano Laredo
4 **Jan. 28–Feb. 3**			
5 **Feb. 4–10**	Feb. 8	**Lecture 2**	J. Brown
6 **Feb. 11–17**			
7 **Feb. 18–24**	*Feb. 18* Feb. 22	*Presidents' day, no classes* **Lecture 3**	 D. Labardini Fragoso
8 **Feb. 25–Mar. 2**			
Mar. 3–9		*Spring break, no classes*	
9 **Mar. 10–16**	Mar. 14	**Lecture 4**	A. Foksha
10 **Mar. 17–23**			
11 **Mar. 24–30**	Mar. 28	**Lecture 5**	S. Gautam
12 **Mar. 31–Apr. 6**			
13 **Apr. 7–13**	Apr. 11	**Lecture 6**	S. Gautam
14 **Apr. 14–20**			

3. Syllabus

Below is a tentative syllabus with topics and references. Drinfeld's ICM paper [Dr] is succint but gives a wonderful panoramic overview of both classical and quantum aspects of the subject.

Lecture 1. Poisson Lie groups and Lie bialgebras (I).
(Poisson manifolds, Poisson Lie groups, Lie bialgebras, Lie's third theorem).
[CP, 1.1,1.2, 1.3A], [ES, 1.4, 2.1, 2.2.1–2.2.2, 2.3.1], [KS, chap. 1, 2.1 and 3].

Lecture 2. Poisson Lie groups and Lie bialgebras (II).
(Manin triples, duals, doubles, examples: complex and compact semisimple Lie groups, dressing action).
[CP, 1.3B, 1.4, 1.5], [ES, 2.2.3, 2.3.2–2.3.3, chap. 4], [Ko, 4.5–4.9], [KS, 2.3, 3.4, chap. 5].

Lecture 3. Classical r–matrices and the CYBE.
[CP, 2.1–2.2], [ES, chap. 3], [Ko, chap. 2], [KS, chap. 4].

Lecture 4. Classical integrable systems.
Main ref.: [STS, sect. 1–5]. Secondary refs.: [Ko, sect. 3], [CP, chap. 2.3].

Lecture 5. Constant solutions of the CYBE.
[ES, chap. 5], [BD].

Lecture 6. Solutions of the CYBE with spectral parameter.
[ES, chap. 6–7], [BD].