DIFFERENTIAL EQUATIONS AND QUANTUM GROUPS NORTHEASTERN UNIVERSITY, SPRING 2010

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The aim of this course will be to understand how quantum groups, such as the one associated to $n \times n$ matrices, can be used to compute the monodromy, or analytic continuation of solutions of certain differential equations in the complex domain.

These equations go under various names: KZ (for Knizhnik–Zamolodchikov), Cherednik–KZ and Casimir and appear in a large number of contexts: low-dimensional topology (knot invariants for example), physics (string theory and statistical mechanics) representation theory and, more recently in algebraic geometry (stability conditions and quantum cohomology). Their knowledge will not be assumed, in fact we shall spend some time studying them per se, nor will the above contexts, which are mentioned merely give a sense of perspective.

The course will only assume familiarity with semisimple Lie algebras and will be a blend of representation theory (quantum groups), some basic complex algebraic geometry (blowups), some category theory (tensor categories) and some deformation theory (Hochschild cohomology).

It will meet on Tuesdays and Thursdays, from 5:50 to 7:20 in 544 NI, starting on January 19.

Following an idea of A. Zelevinsky, the course will end on a mini–conference where course participants will present results related to, but not contained in the course. A syllabus, and a preliminary selection of topics for the mini–conference follow.

1. Syllabus

1.1. **Differential equations on hyperplane complements.**

Holonomy equations of a hyperplane complement.
Main examples:

(1) The Cherednik–KZ (CKZ) connection of a finite reflection group.
(2) The KZ connection of a semisimple Lie algebra.
(3) The Casimir connection of a semisimple Lie algebra.

Focus on the CKZ connection: the braid group and Hecke algebra of a finite reflection group.

1.2. **Hochschild cohomology of an algebra.**

Deformations of an associative algebra A and of its representations.
The Hochschild complex of A.
Applications:

(1) Rigidity of the Hecke algebra of a finite reflection group.
(2) Computation of the monodromy of the CKZ connection.

1.3. **Quantum groups.**

The Drinfeld–Jimbo quantum group $U_\hbar\mathfrak{g}$ of a semisimple Lie algebra $\mathfrak{g}$.
The universal R–matrix of $U_\hbar\mathfrak{g}$.
The quantum Weyl group operators of $U_\hbar\mathfrak{g}$.

1.4. **Braided tensor categories and quasi–triangular quasi–hopf algebras.**
Braided tensor categories.
Hopf algebras, quasi–Hopf algebras, quasi–triangular quasi–Hopf algebras.
Braid group representations.

1.5. **KZ equations and $U\mathfrak{g}$ as a quasi–triangular, quasi–Hopf algebra.**
De Concini–Procesi compactification of hyperplane complements.
Computation of monodromy in terms of local monodromies and associators.
The case of KZ equations: Drinfeld's associator.

1.6. **The Drinfeld–Kohno theorem.**
$U_\hbar\mathfrak{g}$ as a trivial deformation of $U\mathfrak{g}$.
The Hochschild complex of a coalgebra.
Rigidity of quasi–triangular, quasi–Hopf structures on $U\mathfrak{g}$.
The Drinfeld–Kohno theorem.

1.7. **Quasi–Coxeter algebras and Dynkin diagram cohomology.**
D–algebras and quasi–Coxeter algebras.
Deformation theory: the Dynkin complex of a D–algebra.
Quasi–Coxeter, quasi–triangular quasi–bialgebras.
The Dynkin–Hochschild complex of a D–bialgebra.

1.8. **Monodromy of the Casimir connection.**
$U\mathfrak{g}$ as a quasi–Coxeter, quasi–triangular, quasi–bialgebra.
Rigidity of the Hochschild–Dynkin complex of $U\mathfrak{g}$.
Monodromy of the Casimir connection.

2. Preliminary list of topics for the mini–conference

2.1. **Coincidence of various flat connections** [TL1, TL2]**.**

(1) CKZ for the symmetric group and KZ for $GL(n)$.
(2) KZ for $GL(k)$ and Casimir for $GL(n)$.
(3) Casimir equations for $\mathfrak{g}$ and CKZ equations for the Weyl group of $\mathfrak{g}$.

2.2. **Trigonometric connections** [Ch2, Lo, TL4]**.**

(1) General form
(2) Examples:
    (a) the trigonometric KZ connection.
    (b) the affine CKZ connection and degenerate Hecke algebras.
    (c) the trigonometric Casimir connection.

2.3. **Monodromy of the affine CKZ connection** [Ch1]**.**

(1) Monodromy in rank 1 via the hypergeometric equation.
(2) Rank 1 reduction in the general case.
(3) Degeneration to the rational CKZ connection for W classical.

2.4. **Trigonometric Dunkl operators** [He]**.**

(1) Definition of the operators, commutativity and W–equivariance.
(2) Relevance of the degenerate affine Hecke algebra H'.
(3) The centre of H' and quantum integrable systems.
(4) Eigenvalue equations and the trigonometric CKZ equations.

2.5. **Existence of a rational associator** [Dr]**.**

References

[Ch1] I. Cherednik, *Affine extensions of Knizhnik–Zamolodchikov equations and Lusztig's isomorphisms*, Special functions (Okayama, 1990), 63–77, ICM-90 Satell. Conf. Proc., Springer, 1991.

[Ch2] I. Cherednik, *Double affine Hecke algebras*, London Mathematical Society Lecture Note Series, 319. Cambridge University Press, 2005.

[DCP] C. De Concini, C. Procesi, *Wonderful models of subspace arrangements*, Selecta Math. (N.S.) **1** (1995), 459–494. *Hyperplane arrangements and holonomy equations*, Selecta Math. (N.S.) **1** (1995), 495–535.

[Dr] V. G. Drinfeld, *On quasitriangular quasi–Hopf algebras and on a group that is closely connected with* Gal($\overline{Q}/Q$), Leningrad Math. J. **2** (1991), 829–860.

[He] G. Heckman, *Dunkl operators*, Séminaire Bourbaki, Vol. 1996/97. Astérisque No. **245** (1997), Exp. No. 828, 223–246.

[Ks] C. Kassel, *Quantum groups.* Graduate Texts in Mathematics, 155. Springer–Verlag, New York, 1995.

[Lo] E. Looijenga, *Arrangements, KZ systems and Lie algebra homology*, Singularity theory (Liverpool, 1996), 109–130, London Math. Soc. Lecture Note Ser., 263, Cambridge Univ. Press, 1999.

[SS] S. Shnider, S. Sternberg, *Quantum groups. From coalgebras to Drinfel'd algebras. A guided tour.* Graduate Texts in Mathematical Physics, II. International Press, 1993.

[TL1] V. Toledano Laredo, *A Kohno–Drinfeld Theorem for Quantum Weyl Groups*, Duke Mathematical Journal **112** (2002), 421–451.

[TL2] V. Toledano Laredo, *Flat Connections and Quantum Groups*, Acta Applicandae Mathematicae **73** (2002), 155–173.

[TL3] V. Toledano Laredo, *Quasi–Coxeter Algebras, Dynkin Diagram Cohomology and Quantum Weyl Groups*, International Mathematics Research Papers 2008.

[TL4] V. Toledano Laredo, *The trigonometric Casimir connection of a simple Lie algebra*, forthcoming.