

**QUANTUM GROUPS AND STATISTICAL MECHANICS
NORTHEASTERN UNIVERSITY, SPRING 2011**

VALERIO TOLEDANO LAREDO

The aim of this course will be to illustrate how quantum groups (which are deformations of the Lie algebra of $n \times n$ matrices) and their representation theory can be used to better understand models of statistical mechanics which describe particles lying on a 1 or 2-dimensional lattice.

The course will be elementary, in particular we will only consider the quantum groups corresponding to 2×2 matrices (surprisingly perhaps, that gets one pretty far!). Their knowledge won't be assumed, though familiarity with some representation theory of the Lie algebra $\mathfrak{sl}(2)$ will (classification of irreducible representations, Clebsch–Gordan rules).

On the statistical mechanical front, no prior knowledge will be assumed. I will introduce the relevant models (for reference these are called the XXZ model and the six-vertex model) and concepts as we go.

It will meet on Wednesdays, from 4:15 to 5:45 and Thursdays, from 5 to 6:30 in 511 NI, starting on January 20.

1. SYLLABUS

1.1. **The Lie algebra \mathfrak{sl}_2 [H].**

- (1) Presentation.
- (2) Invariant bilinear form and Casimir operator.
- (3) Classification of irreducible representations.
- (4) Weyl character formula.
- (5) Clebsch–Gordan rules.

1.2. **The affine Lie algebra $\widehat{\mathfrak{sl}}_2$ [K].**

- (1) The central extension $\widetilde{\mathfrak{sl}}_2$.
- (2) The affine Lie algebra $\widehat{\mathfrak{sl}}_2$.
- (3) Kac–Moody presentation of $\widehat{\mathfrak{sl}}_2$.
- (4) Classification of irreducible highest weight modules.

1.3. **The quantum group $U_q\mathfrak{sl}_2$ [J, CP].**

- (1) Definition of $U_q\mathfrak{sl}_2$.
- (2) Classification of finite-dimensional representations.
- (3) The universal R -matrix and the quantum Yang–Baxter equations.

1.4. **The quantum affine algebra $U_q\widehat{\mathfrak{sl}}_2$ [CP, CP2, ?].**

- (1) Kac–Moody presentation of $U_q\widehat{\mathfrak{sl}}_2$ and $U_q(L\mathfrak{sl}_2)$.
- (2) Drinfeld's new realisation of $U_q(L\mathfrak{sl}_2)$.
- (3) Classification of irreducible finite-dimensional representations of $U_q(L\mathfrak{sl}_2)$.

1.5. **Introduction to Statistical Mechanics.** Guest lecture by Prof. Chris Beasley.

1.6. **The one-dimensional Ising model** [B].

- (1) The transfer matrix.
- (2) Partition function, free energy and correlation length.
- (3) 'Criticality' of the 1-d Ising model.

1.7. **The two-dimensional Ising model** [B].

- (1) 2-d Ising model on a square lattice.
- (2) High and low temperature expansions of the partition function.
- (3) Kraamers–Wannier duality and critical temperature.
- (4) 2-d Ising model on the triangular and honeycomb lattices.
- (5) Star–triangle relations and triangular–honeycomb duality.
- (6) Critical temperature.

1.8. **The two-dimensional Ising model via commuting transfer matrices**[B].

- (1) Diagonal–to–diagonal transfer matrices.
- (2) 'Star-triangle' relations and commuting transfer matrices. Operator interpretation: the Yang–Baxter equations.
- (3) Functional relations for and determination of the eigenvalues of the transfer matrices.

1.9. **The six vertex/Ice-type model**[B].

- (1) Definition of the model.
- (2) Yang–Baxter relations and commuting transfer matrices.
- (3) Solution of the YB relations: R -matrix with a spectral parameter.
- (4) Baxter's Q -operator, functional relations for the eigenvalues and Bethe ansatz equations.
- (5) Bethe ansatz for the eigenvectors.
- (6) Thermodynamic Bethe ansatz.

REFERENCES

- [B] R. Baxter, *Exactly solved models in statistical mechanics*. Academic Press, 1989, available at <http://tprsv.anu.edu.au/Members/baxter/book>.
- [CP] V. Chari, A. Pressley, *A guide to quantum groups*. Cambridge University Press, Cambridge, 1995.
- [CP2] V. Chari, A. Pressley, *Quantum affine algebras*. *Comm. Math. Phys.* **142** (1991), 261–283.
- [H] J. E. Humphreys, *Introduction to Lie algebras and representation theory*. Graduate Texts in Mathematics, Vol. 9. Springer–Verlag, 1972.
- [J] J. C. Jantzen, *Lectures on quantum groups*. Graduate Studies in Mathematics, 6. American Mathematical Society, 1996.
- [K] V. Kac, *Infinite-dimensional Lie algebras*. Third edition. Cambridge University Press, 1990.