Do not write in the boxes immediately below.

problem	1	2	3	4	5	6	7	8	9	total
points										

MTH1125 Final Exam

December 10, 2002

Instructor's name____________________________ Your name____________________________

Show all work. Answers straight from your calculator are worth zero points.
The number of total points on the exam is 100.

1) (10 points) Find the explicit particular solution to the initial value problem: $(x^2+1)\dfrac{dy}{dx} = 2xy^2; \quad y(1) = 2.$

2) (12 points) A mass of 3 kg is falling through the atmosphere. The mass is acted on by two forces: gravity, which acts straight down, and air resistance which acts straight up. Assume that the force of air resistance is proportional to the velocity, v, of the mass, where the proportionality constant is 6 $N/(m/s)$. Suppose that the initial velocity of the mass is 0, and that "down" has been chosen as the positive direction. Find the velocity of the mass at time $t = 10$ seconds. (Assume that the gravitational constant g is 9.8 m/s^2.)

3) (10 points) Suppose that initially Fred has $500,000$ hairs on his head. The number of hairs, H, on Fred's head changes for two reasons: new hairs grow in, and hair falls out. Suppose that Fred grows new hairs at a constant rate of 50 per day, but Fred loses 0.001% of his hair each day. Write the appropriate differential equation for the rate of change of the number of hairs on Fred's head, and include any initial data, i.e., write the appropriate initial value problem. **DO NOT SOLVE THIS INITIAL VALUE PROBLEM.YOU ARE SIMPLY SUPPOSED TO TRANSLATE THIS WORD PROBLEM INTO A PURE MATHEMATICS PROBLEM.**

4) (10 points) Use Euler's method with step size 0.1 to estimate $y(0.2)$, where y is the solution to the initial value problem $y' = x^2 + y^2, y(0) = 1$. All calculations should be carried out to 3 decimal places.

5) Suppose that the population, P of a certain island changes at a continuous rate given by

$$\frac{dP}{dt} = (0.000001)(P - 1000)(10,000 - P),$$

where t is measured in years. Note that this equation is autonomous.

a) (4 points) If the initial population of the island is 1000 people, what will the population be five years later? Explain. (Hint: Do **not** solve the differential equation! There is a quick solution to this problem.)

b) (7 points) If the initial population of the island is 5000 people, describe qualitatively what happens to the population as times goes on. You must provide some justification for your answer (some formulas, or a phase diagram, or something).

YOU MAY USE THE FOLLOWING POWER SERIES IN THE REMAINING PROBLEMS.

For all x,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$$

For $-1 < x < 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \ldots$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots$$

For $-1 < x \leqslant 1$,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \ldots$$

6) (6 points each) In each part of this problem, find a formula for the n-th partial sum S_n and then determine whether the series converges or diverges. If the series converges, find its sum.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n} = 2 + \frac{2^2}{5} + \frac{2^3}{5^2} + \frac{2^4}{5^3} + \ldots.$

b) $\displaystyle\sum_{n=1}^{\infty} \left[\frac{1}{n+2} - \frac{1}{n+3}\right].$

7) (8 points) Does the series $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!}$ converge or diverge? If it converges, give a simple expression (**not** a decimal approximation) for the sum of the series. (Hint: You **cannot** find a nice formula for the partial sums. Think of series that you know.)

8)

a) (7 points) Give a power series (at least the first four non-zero terms) centered at 0 which equals $\frac{1}{1+x^2}$, and include the interval of convergence. You **must** show some work.

b) (7 points) Recall that $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x)+C$. Give a power series (at least the first four non-zero terms) centered at 0 which equals $\tan^{-1}(x)$, and include the interval of convergence. There should be no arbitrary constant left in your final answer. You **must** show some work.

9)

a) (5 points) Give a power series (at least the first four non-zero terms) centered at 0 which equals $(1+5x)^{0.2}$, and include the interval of convergence.

b) (5 points) Fill in the blanks.

When x is close to 0, the function

$$(1+5x)^{0.2} - 1 - x$$

is approximately equal to the function cx^m where $c =$ ____________ (some constant) and $m =$ ____________ (some positive integer exponent).

c) (3 points) Explain why we need for x to be close to 0 in part b). (Yes - you have to write some words.)