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Research Statement

I am interested in representation theory, combinatorics, and algebraic geometry. My immediate research is on various aspects of cluster algebras. In Section 1, we recall the definition and some basic properties of cluster algebras. In Section 2, we summarize the results in [28] and [29] regarding geometric realization for cluster algebras of finite types and a combinatorial formula for generalized minors in the classical type. Finally, in Section 3, I list some possible projects for future research.

1. Overview

The original motivation for the theory of cluster algebras, first introduced and studied by S. Fomin and A. Zelevinsky in [11], lay in the desire to create an algebraic framework for total positivity and (dual) canonical bases in semisimple algebraic groups ([9], [22], [23]). Examples of cluster algebras including the homogeneous coordinate rings of Grassmannians, Schubert varieties, and other related varieties (after a minor adjustment). Since its inception, the theory of cluster algebras has found many connections and applications: Poisson geometry, Teichmüller theory, discrete integrable systems, quiver representations, preprojective algebras, Calabi-Yau algebras and categories, etc. We refer to the survey papers [4], [5], [7], [13], [15], [18], [30], [31] and to the cluster algebras portal [6] for more information on cluster algebras and their links with other parts of mathematics.

Cluster algebras are a class of constructively defined commutative rings equipped with a distinguished set of generators (*cluster variables*) grouped into overlapping subsets (*clusters*) of the same finite cardinality (the *rank* of a cluster algebra). The generators and the algebraic relations among them are not given at once but are produced by an iterative process of *seed mutations*.

For simplicity, we restrict ourselves to a special class of cluster algebras called *cluster algebras of geometric type*. Let m and n be two positive integers such that $m \geq n$. Let $\mathscr{F}$ be the field of rational functions over the $\mathbb{Q}$ in m independent variables. We use $[1,n]$ to denote the set $\{1,2,\ldots,n\}$. A (labeled) *seed* (of geometric type) in $\mathscr{F}$ is a pair $(\tilde{\mathbf{x}},\tilde{B})$, where $\tilde{\mathbf{x}} = \{x_1,\ldots,x_m\}$ is a set of algebraically independent generators for $\mathscr{F}$, and $\tilde{B} = (b_{i,j})$ is an $m \times n$ integer matrix such that the $n \times n$ submatrix $B = (b_{i,j})$ for $i,j \in [1,n]$ is *skew-symmetrizable*. That is $d_i b_{i,k} = -d_k b_{k,i}$ for some positive integers d_i, where $i,k \in [1,n]$. We refer to $\mathbf{x} = \{x_1,\ldots,x_n\}$ as the *cluster*, to its elements $x_1,\ldots,x_n$ as *cluster variables* (of the seed $(\tilde{\mathbf{x}},\tilde{B})$), to $\mathbf{c} = \{x_{n+1},\ldots,x_m\}$ as the *coefficient tuple*, and to B as the *exchange matrix*.

Let $(\tilde{\mathbf{x}},\tilde{B})$ be a seed in $\mathscr{F}$, and $k \in I$. The *seed mutation* in direction k transforms $(\tilde{\mathbf{x}},\tilde{B})$ into another seed $(\tilde{\mathbf{x}}',\tilde{B}')$, where $\tilde{\mathbf{x}}' = \tilde{\mathbf{x}} - \{x_k\} \cup \{x'_k\}$ whereas $x'_k \in \mathscr{F}$ is determined by the *exchange relation* of the form

$$x_k x'_k = p^+ M^+ + p^- M^- \tag{1.1}$$

Here p^+ and p^- are two monomials in $x_{n+1},\ldots,x_m$ while M^+ and M^- are two monomials in the elements of $\mathbf{x} - \{x_k\}$. The exponents in those monomials are encoded by the matrix $\tilde{B}$. Note that during the mutation process we only *exchange* the cluster variables and leave the coefficient tuple unchanged. Let $\mathbb{Z}[\mathbf{c}^{\pm 1}]$ be the ring of Laurent polynomials in the variables from $\mathbf{c}$. The cluster algebra $\mathscr{A} = \mathscr{A}(\tilde{\mathbf{x}},\tilde{B}) = \mathscr{A}(\mathbf{x},\tilde{B})$ (of rank n) with an initial seed $(\tilde{\mathbf{x}},\tilde{B})$ is the $\mathbb{Z}[\mathbf{c}^{\pm 1}]$-subalgebra of the ambient field $\mathscr{F}$ generated by all cluster variables, that is by the union of all clusters obtained from the initial cluster by iterating seed mutations in all directions.

One surprising result shown in [11] is the *Laurent phenomenon* : Any cluster variable viewed as a rational function in the variables of any given cluster is a Laurent polynomial whose coefficients are integer Laurent polynomials in the coefficient variables $x_{n+1},\ldots,x_m$. This means that even though the numerators of these Laurent polynomials may contain a huge number of monomials, at every stage of the exchange process – moving the numerator for a cluster variable x into the denominator when we compute the cluster variable x' obtained from x by an exchange relation – a cancelation will inevitably occur, leaving a single monomial in the denominator.

It was shown in [14] that with any exchange matrix one can associate an important system of *principal* coefficients which in a certain sense controls all the other choices of coefficients. For a cluster algebra with principal coefficients at an initial cluster $\mathbf{x}$, one can associate with each cluster variable z its $\mathbf{g}$-vector $\mathbf{g}_{z;\mathbf{x}} \in \mathbb{Z}^n$ and the F-polynomial $F_{z;\mathbf{x}} \in \mathbb{Z}[t_1, \ldots, t_n]$. According to [14, Corollary 6.3], to express any cluster variable in a cluster algebra with arbitrary coefficients in terms of an initial cluster, one only needs to know the $\mathbf{g}$-vector and the F-polynomial. The F-polynomials are conjectured (and in many cases proved) to have positive coefficients.

2. Geometric Realization for Cluster Algebras of the Finite Types and Combinatorial Expressions for F-polynomials in Classical Types

The theory of cluster algebras has a lot in common with the theory of Kac-Moody algebras. In both instances, the structure of an algebra in question is encoded by a square integer matrix: a generalized Cartan matrix A in the Kac-Moody case, and an exchange matrix B in the case of cluster algebras. (Note an important distinction between the two cases: the sign pattern of matrix entries is symmetric for A but skew-symmetric for B.) In both instances, there is a natural notion of finite type. We say that a cluster algebra is of *finite type* if it has finitely many (distinct) cluster variables. As shown in [12], the cluster algebras of finite types are classified by the same Cartan-Killing types as semisimple Lie algebras. This occasionally leads to an intriguing collision of two types of symmetry: the coordinate ring of a variety associated with a semisimple algebraic group G may carry a natural cluster algebra structure whose Cartan-Killing type is completely different from that of G. For example, the base affine space G/N for $G = \mathrm{SL}_5$ inherits the symmetry of type A_4 from G, but the coordinate ring $\mathbb{C}[G/N]$ has a natural cluster algebra structure of type D_6. (here we extend the scalars of the cluster algebra from $\mathbb{Z}$ to $\mathbb{C}$).

One way to relate the two classifications to each other is by exhibiting a uniform construction of a variety associated with a semisimple group G whose coordinate ring naturally carries a cluster algebra structure of the same Cartan-Killing type. Such a construction was given in [1, Example 2.24] with the variety in question being the double Bruhat cell $G^{c,c^{-1}}$, where c is a Coxeter element in the Weyl group of G (here and in the sequel, we assume G to be a simply connected semisimple complex Lie group with rank n).

Together with A. Zelevinsky in [29], we improve the construction introduced in [1] in the following two aspects. First, there are many non-isomorphic cluster algebras of the same type differing from each other by the choice of a coefficient system. From this perspective, there is nothing especially distinguished about the coefficient system for $\mathbb{C}[G^{c,c^{-1}}]$. we showed that one can realize the cluster algebra of finite type with principal coefficients at an arbitrary acyclic initial cluster by replacing the double cell $G^{c,c^{-1}}$ with its *reduced* version $L^{c,c^{-1}}$ introduced in [2].

Second and perhaps more importantly, in [1] only the initial cluster in $\mathbb{C}[G^{c,c^{-1}}]$ was given explicitly, while no information was obtained about the rest of the cluster variables. In [29], we computed explicitly all cluster variables in $\mathbb{C}[L^{c,c^{-1}}]$, and they turn out to be an interesting special family of principal generalized minors. Generalized minors, first introduced in [9] for the study of total positivity in a simply connected semisimple complex algebraic group G, are a special family of regular functions $\Delta_{\gamma,\delta}$ on G. These functions are suitably normalized matrix coefficients corresponding to pairs of extremal weights (γ, δ) in some fundamental representation of G. We call a generalized minor $\Delta_{\gamma,\delta}$ *principal* if $\gamma = \delta$. We summarize some of the main results from [29] in the following two theorems.

Theorem 2.1 ([29], Theorem 1.4 and Theorem 1.10). *Let $c = s_{i_1} \cdots s_{i_n}$. Then the cluster variables in $\mathbb{C}[L^{c,c^{-1}}]$ are parametrized by a special set of extremal weights $\{c^m\omega_k : k \in [1,n], 0 \le m \le h(k;c)\}$ where $h(k;c)$ is the smallest positive integer such that $c^{h(k;c)}\omega_k = w_\circ(\omega_k)$ where $w_\circ$ is the longest element in the Weyl group. Furthermore, identifying $\mathbb{Z}^n$ with the weight lattice by means of the basis $\omega_1, \ldots, \omega_n$ of fundamental weights, the $\mathbf{g}$-vector of the cluster variable corresponding to $c^m\omega_k$ gets identified with $c^m\omega_k$.*

Interestingly, the combinatorics related to the action on weights by the cyclic group $\langle c\rangle$ generated by c in the above theorem turns out to be very close to the one developed in the classical paper [20] by B. Kostant, and used in [19] for (seemingly) completely different purposes. An alternative description of these $\mathbf{g}$-vectors was given in [25, Theorem 10.2]. It is stated in different terms, and proved by a quite different method, relying on one of the conjectures in [14]. It is not difficult to check the equivalence of the two descriptions.

To give a formula for the F-polynomials, recall that, for each $i \in [1, n]$, there are one-parameter root subgroups in G given by

$$x_i(t) = \varphi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad x_{\bar{i}}(t) = \varphi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \tag{2.1}$$

where $\varphi_i : SL_2 \to G$ denotes the canonical embedding corresponding to the simple root α_i.

Theorem 2.2 ([29], Theorem 1.12). *In the above setup, the F-polynomials are given by*

$$F_{c^m\omega_k}(t_1, \dots, t_n) = \Delta_{c^m\omega_k, c^m\omega_k}(x_{\bar{i}_1}(1) \cdots x_{\bar{i}_n}(1) x_{i_n}(t_{i_n}) \cdots x_{i_1}(t_{i_1})). \tag{2.2}$$

For the type $\mathsf{A_n}$ case, the generalized minors specialize to the ordinary minors, and Lindström's Lemma (see [21], [16], [17] and [10]) provides a combinatorial interpretation for the minors of a matrix of certain form (written as a product of the elementary Jacobi matrices). In [28], the author generalized this result to other classical types and proved the positivity of F-polynomials in cluster algebras of classical types. The minors are given in terms of weighted vertex-disjoint paths in a weighted directed graph associated with the Coxeter element c. Here we give an example in the type $\mathsf{D_4}$ case. Let $c = s_1 s_2 s_3 s_4 = (1, 2, 3, \bar{1}, \bar{2}, \bar{3})(4, \bar{4})$, written as a permutation on the index set $[1, 4] \cup [\bar{1}, \bar{4}]$, then $c^2 \cdot [1, 2] = \{3, \bar{1}\}$. Then the F-polynomial $F_{c^2\omega_2}(t_1, t_2, t_3, t_4)$ is equal to the sum of weights of all collections of vertex-disjoint paths in the directed graph $\Gamma(\mathsf{D_n}, s_1 s_2 s_3 s_4)$ (shown in Figure 1) with the sources and sinks labeled by $c^2 \cdot [1, 2]$. The weight of a family of paths is defined as the product of the weights of its edges. (in $\Gamma(\mathsf{D_4}, s_1 s_2 s_3 s_4)$, all unlabeled edges have weight 1.)

$$F_{c^2\omega_2}(t_1, t_2, t_3, t_4) = 1 + t_1 + t_2 + 2t_1t_2 + t_1t_2t_3 + t_1t_2t_4 + t_1t_2^2 + t_1t_2^2t_3 + t_1t_2^2t_4 + t_1t_2^2t_3t_4.$$

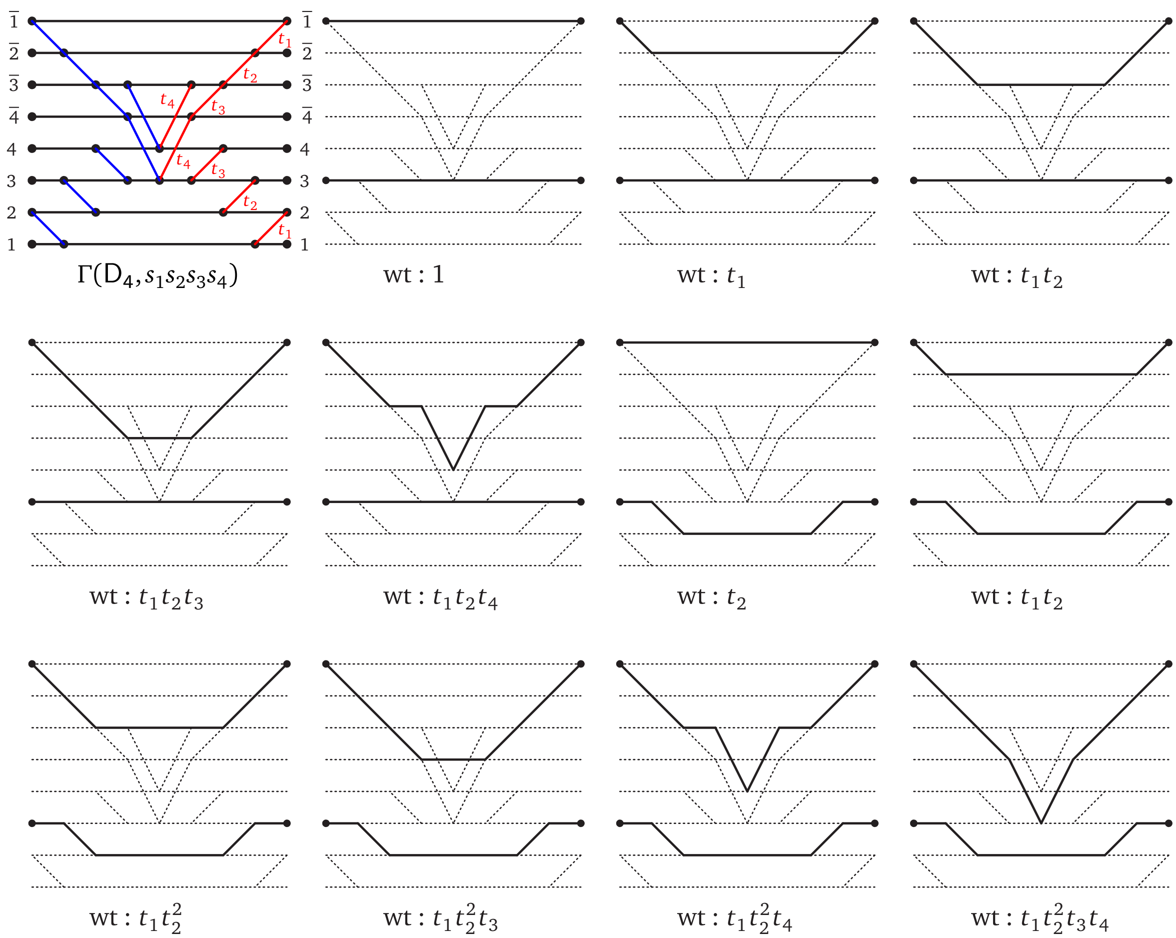

FIGURE 1

3. Future Plans

Geometric realization for cluster algebras beyond the finite types: It is natural to pursue a geometric realization for cluster algebras beyond the finite types with the similar spirit as in [29]. That is realizing them as coordinate rings of some "varieties" naturally associated to Kac-Moody groups.

Quantum cluster algebras: In [3], A. Berenstein and A. Zelevinsky defined the quantum cluster algebras as a certain noncommutative deformation of the cluster algebras, in particular, the quantum double Bruhat cells were studied. I plan to extent the results in [29] to the quantum cluster algebras associated with the quantum reduced double Bruhat cells.

Comparisons of other formulas for F-polynomials: There are other formulas for the F-polynomials and proofs for the positivity conjecture in the literature. In particular, Musiker, Schiffler and Williams's work in [24] deals with cluster algebras from surfaces defined and studied in [8]; the results in [26, 27] by Tran have the same generality as in [28]. The answer given in [28] was in very different terms and obtained by totally different methods. I plan to compare those different formulas.

Orthogonal polynomials: For the type A_n case (i.e., when $G = \mathrm{SL}_{n+1}$), let $c = s_1 \cdots s_n$, in the standard numbering of simple roots. In this case $L^{c,c^{-1}}$ is the subvariety of $\mathrm{SL}_{n+1}(\mathbb{C})$ consisting of tridiagonal matrices of the form

$$M = \begin{pmatrix} v_1 & y_1 & 0 & \cdots & 0 \\ 1 & v_2 & y_2 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & y_n \\ 0 & \cdots & 0 & 1 & v_{n+1} \end{pmatrix} \tag{3.1}$$

with all $y_1, \ldots, y_n$ non-zero. The exchange relations from the initial cluster can be rewritten as follows:

$$x_{[1,k+1]} = v_{k+1} x_{[1,k]} - y_k x_{[1,k-1]} \quad (k = 1, \ldots, n), \tag{3.2}$$

where $x_{[i,j]}$ denote the regular function on M given by the principal minor with rows and columns $i, i+1, \ldots, j$, with the convention that $x_{[i,j]} = 1$ unless $1 \le i \le j \le n+1$. These relations play a fundamental part in the classical theory of orthogonal polynomials in one variable. Thus, this cluster algebra can be viewed as some kind of "enveloping algebra" for this theory. It would be interesting to study the theory of orthogonal polynomials from the perspective of the theory of cluster algebras and extent to other choose of Coxeter elements and other Cartan-Killing types.

Generalizing the classical determinantal calculus to other classical types: There are many classical determinantal identities that can be extended from the special linear group to any simply connected semisimple group, see e.g., Theorems 1.16, 1.17 and Proposition 4.5 in [9]. With the combinatorial interpretation for generalized minors in all classical types given in [28], I hope to investigate which other classical determinantal identities can be generalized to semisimple groups of classical types.

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